On the vanishing of Iwasawa invariants of certain cyclic extensions of $\boldsymbol{Q}$ with prime degree II

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1. Introduction. Throughout the paper, we fix an odd prime number ℓ. For a prime number p congruent to one module ℓ, we denote by k_p the unique subfield of $\boldsymbol{Q}(\zeta_p)$ of degree ℓ, where ζ_p is a primitive p-th root of unity. Let $\boldsymbol{F}_\ell = \boldsymbol{Z}/\ell\boldsymbol{Z}$ and let $(\frac{a}{p})_\ell$ be the ℓ-th power residue symbol for an integer a. In [2], we proved the following theorem.

Theorem 1.1 (Corollary 2.3 in [2]). *Let p and q be distinct prime numbers congruent to one modulo ℓ satisfying $(\frac{\ell}{p})_\ell \neq 1$, $(\frac{p}{q})_\ell \neq 1$, $q \not\equiv 1 \pmod{\ell^2}$. Let $x, y, z \in \boldsymbol{F}_\ell$ such that $(\frac{q\ell^x}{p})_\ell = 1$, $(\frac{\ell p^y}{q})_\ell = 1$ and $pq^z \equiv 1 \pmod{\ell^2}$. If $xyz \neq -1$, then for any subfield k of k_pk_q of degree ℓ, the Iwasawa invariants $\lambda_\ell(k)$ and $\mu_\ell(k)$ are both zero.*

In this paper, we investigate the case $(\frac{p}{q})_\ell = 1$.

2. Theorems. Let p and q be distinct prime numbers congruent to one modulo ℓ. We assume that $p \not\equiv 1 \pmod{\ell^2}$, $q \not\equiv 1 \pmod{\ell^2}$, $(\frac{\ell}{p})_\ell \neq 1$ and $(\frac{q}{p})_\ell = (\frac{p}{q})_\ell = 1$. We treat the case $(\frac{\ell}{q})_\ell = 1$ and the case $(\frac{\ell}{q})_\ell \neq 1$ separately. In the case $(\frac{\ell}{q})_\ell = 1$, we have the following theorem.

Theorem 2.1. *Assume that $(\frac{\ell}{q})_\ell = 1$. Let k be a subfield of k_pk_q of degree ℓ which is different from k_p and k_q. If $p \notin E_k k^{\times\ell}$, then $\lambda_\ell(k)$ and $\mu_\ell(k)$ are both zero.*

Here E_k denotes the unit group of k. In the case $(\frac{\ell}{q})_\ell \neq 1$, we need to specify k explicitly. Let

$$\sigma = \left(\frac{k_p/\boldsymbol{Q}}{\ell}\right),\ \tau = \left(\frac{k_q/\boldsymbol{Q}}{\ell}\right)$$

be Frobenius automorphisms. We identify the Galois group $G(k_p/\boldsymbol{Q})$ with $G(k_pk_q/k_q)$ and $G(k_q/\boldsymbol{Q})$ with $G(k_pk_q/k_p)$ canonically. Then $G(k_pk_q/\boldsymbol{Q}) = <\sigma, \tau>$. If k is a subfield of k_pk_q with degree ℓ which is different from k_p and k_q, then $G(k_pk_q/k) = <\sigma\tau^i>$ for some $i \in \boldsymbol{F}_\ell^\times$. In this case, we have the following theorem.

Theorem 2.2. *Assume that $(\frac{\ell}{q})_\ell \neq 1$. Let k be a subfield of k_pk_q which corresponds to $<\sigma\tau^i>$ for some $i \in \boldsymbol{F}_\ell^\times$ and z the element of $\boldsymbol{F}_\ell^\times$ such that $pq^z \equiv 1 \pmod{\ell^2}$. If $pq^{z/i} \notin E_k k^{\times\ell}$, then $\lambda_\ell(k)$ and $\mu_\ell(k)$ are both zero.*

3. Proof. We shall prove Theorem 2.2. For a Galois extension k of $\boldsymbol{Q}$, we denote by $A(k)$ the ℓ-primary part of the ideal class group of k and $B(k)$ the subgroup of $A(k)$ consisting of elements which are invariant under the action of $G(k/\boldsymbol{Q})$. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$ be the prime ideals of k which are ramified in $k/\boldsymbol{Q}$. If $k/\boldsymbol{Q}$ is a cyclic extension of degree ℓ, then $B(k)$ is an ℓ-elementary abelian group of rank $s-1$ generated by $\mathrm{cl}(\mathfrak{p}_1), \mathrm{cl}(\mathfrak{p}_2), \ldots, \mathrm{cl}(\mathfrak{p}_s)$.

Let $\boldsymbol{Q}_1$ be the subfield of $\boldsymbol{Q}(\zeta_{\ell^2})$ of degree ℓ and put

$$\eta = \left(\frac{\boldsymbol{Q}_1/\boldsymbol{Q}}{q}\right).$$

Then $G(\boldsymbol{Q}_1/\boldsymbol{Q}) = <\eta>$. Let $\mathfrak{p}_p$ (resp. $\mathfrak{p}_q$) be the prime ideal of k lying over p (resp. q). Since $p \not\equiv 1 \pmod{\ell^2}$ and $q \not\equiv 1 \pmod{\ell^2}$, $\mathfrak{p}_p$ and $\mathfrak{p}_q$ inert in $k\boldsymbol{Q}_1/k$. So, if we show that both $\mathfrak{p}_p$ and $\mathfrak{p}_q$ become principal in $k\boldsymbol{Q}_1$, we have $\lambda_\ell(k) = \mu_\ell(k) = 0$ from Corollary 3.6 of [3].

In order to show that both $\mathfrak{p}_p$ and $\mathfrak{p}_q$ become

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