On totally real cubic fields whose unit groups are of type $\{\theta + r, \theta + s\}$

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(Communicated by Shokichi IYANAGA, M. J. A., Dec. 14,1998)

1. Introduction. Let $f(x)$ be a cubic polynomial with rational integer coefficients, which is monic and irreducible. Suppose that all roots θ, θ' and θ'' of $f(x) = 0$ are real, and put $K = \mathbf{Q}(\theta)$. Denote by D_f the discriminant of polynomial $f(x)$. Let $\mathfrak{o}_K$ and E_K be the ring of integers and the group of units of K respectively. Moreover we denote by E_K^+ the subset of E_K consisting of the units ε with $N_{K/\mathbf{Q}}\varepsilon = 1$. It is well known by a theorem of Dirichlet [6] that there exists a system of fundamental units $\{\varepsilon_1, \varepsilon_2\}$ such that

$$E_K = \{\pm 1\} \times E_K^+ \text{ and } E_K^+ = \langle \varepsilon_1, \varepsilon_2 \rangle.$$

Our purpose is to determine totally real cubic fields such that the system of fundamental units can be given in the form $\{\theta + r, \theta + s\}$ for some integers r, s. Note that we can reduce our problem to the case that θ is a unit in K (i.e., $r = 0$).

First, for the minimal polynomial $f(x)$ of θ over $\mathbf{Q}$, we can get the following:

Proposition 1. Suppose that s is a non-zero integer and both θ and $\theta + s$ are in E_K. Then there is an integer t such that

(a) if θ and $\theta + s$ are in E_K^+, then $f(x) = x(x+s)(x+t) - 1$.

(b) if θ and $-\theta - s$ are in E_K^+, then $f(x) = x\left(x^2 + (s+t)x + \left(st - \frac{2}{s}\right)\right) - 1$.

It is easy to prove this proposition.

Conversely we should investigate whether $\{\theta, \theta + s\}$ is a system of fundamental units. As for (i), we can reduce to the case $t \geq 1$, $s \geq t + 1$ because of $\theta(\theta + t) = (\theta + s)^{-1}$. In this condition, Stender [3] and Thomas [4] proved $E_K^+ = \langle \theta, \theta + s \rangle$, but we will prove this in a different way. As for (ii), there are only four cases $s = \pm 1, \pm 2$. The case (ii) $s = 1$ was studied by Watabe [5] completely.

Our main results are as follows:

Theorem 1 (Stender [3], Thomas [4]). In the case $f(x) = x(x+t)(x+s) - 1$ $(s, t \in \mathbf{Z})$, if D_f is positive, square free and $t \geq 1$, $s \geq t + 1$, then $E_K^+ = \langle \theta, \theta + s \rangle$ holds.

Theorem 2 $(s = -1)$. In the case $f(x) = x(x^2 + (t-1)x + (-t+2)) - 1$ $(t \in \mathbf{Z})$, if D_f is positive and square free, then $E_K^+ = \langle \theta, -\theta + 1 \rangle$ holds.

Theorem 3 $(s = 2)$. In the case $f(x) = x(x^2 + (t+2)x + 2t - 1) - 1$ $(t \in \mathbf{Z})$, if D_f is square free, then $E_K^+ = \langle \theta, -\theta - 2 \rangle$ holds.

Theorem 4 $(s = -2)$. In the case $f(x) = x(x^2 + (t-2)x - 2t + 1) - 1$ $(t \in \mathbf{Z})$, if D_f is positive, both of $t + 1$ and $4t^2 + 8t - 23$ are square free and $t \not\equiv 2 \pmod 3$, then $E_K^+ = \langle \theta, -\theta + 2 \rangle$ holds.

2. Preliminaries. We define a function S from E_K to $\mathbf{Z}$ by

$$S(\varepsilon) = \frac{1}{2}\{(\varepsilon - \varepsilon')^2 + (\varepsilon' - \varepsilon'')^2 + (\varepsilon'' - \varepsilon)^2\}.$$

Moreover, define $\mathscr{A}(K)$, and $\mathscr{B}_{\varepsilon_1}(K)$ for ε_1 in $\mathscr{A}(K)$ by

$$\mathscr{A}(K) = \{\varepsilon \in E_K^+ \backslash \{1\} \mid S(\varepsilon) \text{ is minimum}\},$$

$$\mathscr{B}_{\varepsilon_1}(K) = \{\varepsilon \in E_K^+ \backslash \{\varepsilon_1^n ; n \in \mathbf{Z}\} \mid S(\varepsilon) \text{ is minimum}\}.$$

The following lemmas will be useful for the proof of theorems.

Lemma 1 (Brunotte, Halter-Koch [2]). If ε_1 is in $\mathscr{A}(K)$ and ε_2 is in $\mathscr{B}_{\varepsilon_1}(K)$, then $(E_K^+ : \langle \varepsilon_1, \varepsilon_2 \rangle) \leq 4$ holds.

Lemma 2 (Godwin [1]). For any ε, ε_1, ε_2 in E_K^+ and integer $m \geq 2$, we have

$$S(\varepsilon)^2 < 9S(\varepsilon^2),\ S(\varepsilon)^3 < 9S(\varepsilon^3),\ S(\varepsilon)^m < \frac{3^{m+1}}{2}S(\varepsilon^m),$$

$$S(\varepsilon_1\varepsilon_2) < 3S(\varepsilon_1)S(\varepsilon_2),\ S(\varepsilon^{-1}) \leq S(\varepsilon)^2.$$

Lemma 3. In the conditions of Theorem 1, it holds that

$$S(\theta(\theta + s)) \leq S(\theta)^2,\ S(\theta^2(\theta + s)) < S(\theta)^3.$$

Proof. We can easily prove Lemma 3 by elementary calculation. □

Lemma 4. In the conditions of Theorem 1, we have $S(\theta) \geq 12$.

Proof. We have $S(\theta) = (t + s)^2 - 3st = t^2$