An analogue of the Hardy theorem for the Cartan motion group

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(Communicated by Heisuke HIRONAKA, M. J. A., Dec. 14, 1998)

1. Introduction. The aim of this note is to prove an analogue of the Hardy theorem for the Cartan motion group. In the case of the Euclidean space, various forms of the uncertainty principle between a function and its Fourier transform are known. One of such theorems is known as the Hardy theorem. The Hardy theorem (cf. [1, pp. 155-158]) asserts that if a measurable function f on $\boldsymbol{R}$ satisfies $|f| \leq C\exp\{-ax^2\}$ and $|\hat{f}| \leq C\exp\{-by^2\}$ then $f=0$, or f is a constant multiple of $\exp\{-ax^2\}$, or there are infinitely many such f according as $ab > 1/4$, or $ab = 1/4$, or $ab < 1/4$. (Here we take $\hat{f}(y) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(x)\exp\{\sqrt{-1}\,xy\}\,dx$ as the definition of the Fourier transform of f.)

Recently A. Sitaram and M. Sundari [5] generalized this theorem to the cases of the semisimple Lie groups with one conjugacy class of Cartan subgroups, the Riemannian symmetric spaces and $SL(2, \boldsymbol{R})$. And also M. Sundari [6] showed the Hardy theorem for the Euclidean motion group. By the way, in the case of the Cartan motion group, K. Kumahara [4] defined the Fourier transform by using the representation π_ξ induced from $\xi \in \mathfrak{p}^*$ and studied the images of some function spaces under the Fourier transform. By using estimates of matrix elements of π_ξ and applying a similar argument to [5,6], we can get an analogue of the Hardy theorem for the Cartan motion group.

2. Notation and preliminaries. The standard symbols $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ shall be used for the integers, the real numbers and the complex numbers. If V is a vector space over $\boldsymbol{R}$, V_c, V^* and V_c^* denote its complexification, its real dual and its complex dual, respectively. For $v \in V_c$, $\Re v$, $\Im v$ and $\bar{v}$ denote its real part, its imaginary part and its complex conjugate, respectively. For a topological space S and a subset T of S, $\mathrm{Cl}(T)$ denotes the closure of T in S. For a Lie group L, $\hat{L}$ denotes the set of equivalence classes of irreducible unitary representations of L.

Let G_0 be a connected semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. We fix a maximal compact subgroup K of G_0 and denote by θ the corresponding Cartan involution. We set $\mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}$. Let $G_0 = KAN$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be Iwasawa decompositions of G_0 and $\mathfrak{g}$, respectively. We denote by $\langle \cdot, \cdot \rangle$ the Killing form of $\mathfrak{g}$ and put $\|X\|^2 = -\langle X, \theta X \rangle$. We also use the same symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the bilinear form and the norm on $\mathfrak{p}^*$ coming from those on $\mathfrak{p}$. In the notation introduced above, the semidirect product $G = K \ltimes \mathfrak{p}$ is called the Cartan motion group. For any $g \in G$, we write $g = (k(g), X(g))$, where $k(g) \in K$ and $X(g) \in \mathfrak{p}$. If $k \in K$ and $X \in \mathfrak{p}$, we simply write k for $(k, 0)$ and X for (e, X), e being the unit element in K. With these notation, we have

$$(2.1) \quad k(g^{-1}) = k(g)^{-1}, \quad X(g^{-1}) = -\mathrm{Ad}(k(g))^{-1}X(g),$$
$$k(g_1g_2) = k(g_1)k(g_2), \quad X(g_1g_2) = \mathrm{Ad}(k(g_1))X(g_2) + X(g_1),$$

for $g, g_1, g_2 \in G$.

Let M denote the centralizer of $\mathfrak{a}$ in K. For $g \in G$ and $\xi \in \mathfrak{p}^*$, we define $g\xi \in \mathfrak{p}^*$ by $g\xi(X) = \xi(\mathrm{Ad}(k(g))^{-1}X)$, $(X \in \mathfrak{p})$. Let $\mathfrak{a}^+$ and $\mathfrak{a}^{*+}$ be the positive Weyl chambers in $\mathfrak{a}$ and $\mathfrak{a}^*$, respectively. For $X \in \mathfrak{p}$, we choose X^+ in $\mathrm{Cl}(\mathfrak{a}^+)$ such that $X^+ \in \mathrm{Ad}(K)X \cap \mathrm{Cl}(\mathfrak{a}^+)$. And if $\xi \in \mathfrak{p}^*$, we define $\xi^+ \in \mathrm{Cl}(\mathfrak{a}^{*+})$ by a similar way to the case of $\mathfrak{p}$. Let $g \in G$. If we pick $k_1, k_2 \in K$ such that $\mathrm{Ad}(k_1)X(g)^+ = X(g)$ and $k_2 = k_1^{-1}k(g)$, then g is written as $g = k_1X(g)^+k_2$. Thus we have the decomposition $G = K\,\mathrm{Cl}(\mathfrak{a}^+)K$.

Finally let dk be the Haar measure on K normalized as $\int_K dk = 1$. The Killing form induces Euclidean measures on $\mathfrak{p}$ and $\mathfrak{p}^*$. We nor-

*) Partly supported by the Grants-in-Aid for Scientific Research (No. 09640190).

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