The semilattices of nilextensions of left groups and their varieties

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(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1998)

Abstract: In this paper, we characterize the semigroups which are the semilattices of nilextensions of left groups and prove that these semigroups form a variety defined by the equation $(xy)^{\omega}(yx)^{\omega} = (xy)^{\omega}$. Moreover, we obtain some decompositions of this variety by Mal'cev product and semidirect product. In particular, we prove that this variety is just the semidirect product **G*R** of the variety **G** of groups and the variety **R** of $\mathcal{R}$-trivial semigroups.

1. Introduction and preliminaries. A semigroup S is called a semilattice Y of semigroups of type T if there is a homomorphism Ψ from S onto a semilattice Y and the inverse image of each element of Y under Ψ is a semigroup of type T. In [3], Davenport studied the semigroups which are semilattices of nilextensions of groups. The purpose of this paper is to characterize the class of finite semigroups that are semilattices of nilextensions of left groups and to generalize some results of [3]. For simplicity's sake, we denote by **SNLG** the class of finite semigroups which are semilattices of nilextensions of left groups.

First, we will characterize the semigroups in **SNLG** and prove that **SNLG** is defined by equation $(xy)^{\omega}(yx)^{\omega} = (xy)^{\omega}$. Hence **SNLG** is a variety in the sense of Eilenberg [4] since it is closed under finite products, subsemigroups and homomorphic images. Then, using the functorially minimal L' homomorphic image, a concept introduced in [1], we give a decomposition of **SNLG** by semidirect product **G*R**. We also consider some decompositions of **SNLG** by Mal'cev products.

For convenience let us review some definitions and facts germane to what follows. If S is a semigroup, then S^1 is the semigroup obtained by adjoining an identity element 1 to S. A semigroup S is said to be archimedean, if for any $a, b \in S$, there exist $m, n \in N$ such that $a^m \in S^1bS^1$, $b^n \in S^1aS^1$.

Lemma 1.1[8]. A semigroup S is a semilattice of archimedean semigroups if and only if, for every $a, b \in S$, the condition $b \in S^1aS^1$ implies $b^i \in S^1a^2S^1$ for some positive integer i.

A finite semigroup is a left group, if for any $x \in S$, $S = Sx$.

Lemma 1.2 [Theorem 1.27,2]. Let S be a semigroup. Then the following conditions are equivalent:

(1) S is a left group;

(2) S is a regular semigroup whose idempotent elements form a left zero semigroup;

(3) S is isomorphic to a direct product of a left zero semigroup and a group.

Let I be an ideal of S. S is a nilextension of I if for any $a \in S$, there exists a positive integer n such that $a^n \in I$.

Throughout this paper, all semigroups are finite.

For any element x in a semigroup S, we denote by x^{ω} the unique idempotent element in the subsemigroup of S generated by x.

A semigroup S is $\mathcal{R}$-trivial, if S satisfies the equation $(xy)^{\omega}x = (xy)^{\omega}$.

For a semigroup S, we denote by $E(S)$ the set of idempotents of S.

For non defined notations and terminology, the reader is referred to [5], [7], [9].

2. The main structure theorem.

Theorem 2.1. Let S be a semigroup. Then the following are equivalent:

(1) $S \in$ **SNLG**;

(2) Each regular $\mathcal{D}$-class of S is a left group;

(3) For all x and y in S, if xy and yx are idempotent, then $(xy)(yx) = xy$;

(4) S satisfies the equation $(xy)^{\omega}(yx)^{\omega} =$