An explicit description of positive Riesz distributions on homogeneous cones

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Introduction. Riesz distributions, originally introduced by M. Riesz [6] on the Lorentz cone, are the analytic continuation of the distribution defined by a relatively invariant measure on a homogeneous cone. In general, Riesz distributions are compositions of complex measures supported by the closure of the cone with differential operators. Gindikin [3] describes when the Riesz distribution is a positive measure. Since positive Riesz distributions on symmetric cones are closely connected with the analytic continuation of holomorphic discrete series representations of semisimple Lie groups as is shown by Vergne and Rossi [8], one can determine the so-called Wallach set from the result of Gindikin (see [1, p. 288] for this). Therefore we shall call the positivity set for the Riesz distributions *the Gindikin-Wallach set.* In the present paper, we show that the structure of the Gindikin-Wallach set can be understood clearly by relating it to the orbit structure of the closure of the cone. Moreover we give an explicit description to each of the positive Riesz distributions as a measure on an orbit in the closure of the cone.

1. Preliminaries. Our study is based on the structure theory of normal j-algebras developed in [5]. Here we recall the definition of normal j-algebras. Let $\mathfrak{g}$ be a real split solvable Lie algebra, j a linear mapping on $\mathfrak{g}$ such that $j^2 = -\mathrm{id}_{\mathfrak{g}}$, ω a linear form on $\mathfrak{g}$. The triple $(\mathfrak{g}, j, \omega)$ is called a *normal j-algebra* if the following (i), (ii) are satisfied: (i) $[Y, Y'] + j[Y, jY'] + j[jY, Y'] - [jY, jY'] = 0$ for all $Y, Y' \in \mathfrak{g}$, (ii) $(Y|Y')_\omega := \langle [Y, jY'], \omega \rangle$ defines a j-invariant inner product on $\mathfrak{g}$. We assume throughout this paper that our normal j-algebra $(\mathfrak{g}, j, \omega)$ corresponds to a Siegel domain of tube type. Let $\mathfrak{a}$ be the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ relative to $(\cdot|\cdot)_\omega$. Then $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$. Let $r := \dim \mathfrak{a}$.

Proposition 1 ([5, Chapter 1, Sections 3 and 5]). (i) *There is a linear basis $\{A_1, \ldots, A_r\}$ of $\mathfrak{a}$ such that if one puts $E_l := -jA_l$, then $[A_k, E_l] = \delta_{kl} E_l$ $(1 \le k, l \le r)$.*
(ii) *Let $\alpha_1, \ldots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $A_1, \ldots, A_r$. Then $\mathfrak{g} = \mathfrak{h} \oplus V$ with*

$$(1) \qquad \mathfrak{h} = \mathfrak{a} \oplus \Big(\sum_{1 \le k < m \le r}^{\oplus} \mathfrak{g}_{(\alpha_m - \alpha_k)/2} \Big),$$

$$(2) \quad V = \Big(\sum_{k=1}^{r}{}^{\oplus} \mathbf{R} E_k \Big) \oplus \Big(\sum_{1 \le k < m \le r}^{\oplus} \mathfrak{g}_{(\alpha_m + \alpha_k)/2} \Big),$$

where $\mathfrak{g}_\alpha := \{Y \in \mathfrak{g} \mid [C, Y] = \alpha(C) Y$ for all $C \in \mathfrak{a}\}$ for $\alpha \in \mathfrak{a}^*$.
(iii) *One has $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, V] \subset V$, and $[V, V] = \{0\}$.*

Let H be the simply connected Lie group corresponding to $\mathfrak{h}$. Then H acts on V by the adjoint action.

Lemma 2 ([7, Theorem 4.15]). *Put $E := \sum_{k=1}^{r} E_k \in V$ and let Ω be the H-orbit in V through E. Then Ω is an open convex cone in V containing no line, and H acts on Ω simply transitively.*

According to (1), we express every $T \in \mathfrak{h}$ by $T = \sum_{k=1}^{r} t_{kk} A_k + \sum_{m>k} T_{mk}$ $(t_{kk} \in \mathbf{R}, T_{mk} \in \mathfrak{g}_{(\alpha_m - \alpha_k)/2})$. Let Π be the open subset $\{T \in \mathfrak{h} \mid t_{kk} > 0$ for all $k = 1, \ldots, r\}$ of $\mathfrak{h}$. Putting $T_{kk} := (2 \log t_{kk}) A_k$ $(1 \le k \le r)$ and $L_k := \sum_{m>k} T_{mk}$ $(1 \le k \le r-1)$ for $T \in \Pi$, we set $\gamma(T) := \exp T_{11} \cdot \exp L_1 \cdot \exp T_{22} \cdots \exp L_{r-1} \cdot \exp T_{rr}$. Then γ is a diffeomorphism from Π onto H. Using this γ, we have the following multiplication formula for the elements of H.

Proposition 3. *For $T, T' \in \Pi$, one has $\gamma(T)\gamma(T') = \gamma(T'')$ with*

$$t''_{kk} = t_{kk} t'_{kk} \ (1 \le k \le r),$$
$$T''_{mk} = t_{mm} T'_{mk} + \sum_{k<l<m} [T_{ml}, T'_{lk}] + t'_{kk} T_{mk} \qquad (1 \le k < m \le r).$$

2. Orbit structure of $\overline{\Omega}$. Accordind to (2), we decompose every $x \in V$ as $x = \sum_{k=1}^{r} x_{kk} E_k + \sum_{m>k} X_{mk}$ $(x_{kk} \in \mathbf{R}, X_{mk} \in \mathfrak{g}_{(\alpha_m + \alpha_k)/2})$. Define $E^* \in \mathfrak{g}^*$ by $\langle x + T, E^* \rangle = \sum_{k=1}^{r} x_{kk}$ $(x \in V,$