Special values of zeta functions of the simplest cubic fields and their applications

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1. Preliminaries. In [2], Halbritter and Pohst computed the values of partial zeta functions of totally real cubic fields. In this paper, applying their results to the simplest cubic fields, we explicitly compute some special values of partial zeta-functions of these fields. And as applications, we give a necessary condition for class numbers of the simplest cubic fields to be 1 and construct the simplest cubic fields with class numbers divisible by a given rational integer n.

First we restate the main theorem of [2]. (The meaning of notations such as $\binom{6}{m_1,\, m_2}$, $B(3, m_1, m_2, 6-(m_1+m_2), (E_\nu B_\rho)^*, 0)$ will be explained in Remarks 1,2 after the statement of the theorem.)

Theorem 1.1 (*Halbritter and Pohst*). *Let K be a totally real cyclic cubic field with discriminant Δ. For $\alpha \in K$ the conjugates are denoted by α' and α'', respectively. Furthermore, for $\alpha \in K$, let $Tr(\alpha) := \alpha + \alpha' + \alpha''$ and $N(\alpha) := \alpha\alpha'\alpha''$. Let $\{\varepsilon_1, \varepsilon_2\}$ be a system of fundamental units of K. Define L by $L := \ln|\varepsilon_1/\varepsilon''_1| \ln|\varepsilon'_2/\varepsilon''_2| - \ln|\varepsilon'_1/\varepsilon''_1| \ln|\varepsilon_2/\varepsilon''_2|$. Let W be an integral ideal of K with basis $\{\omega_1, \omega_2, \omega_3\}$. Let $\rho = \tilde{\omega}_3$ for a dual basis $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3$ of W subject to*

$$Tr(\omega_i\tilde{\omega}_j) = \delta_{ij} \quad (1 \le i, j \le 3).$$

For $j = 1, 2$, set

$$E_j = \begin{pmatrix} 1 & 1 & 1 \\ \varepsilon_j & \varepsilon'_j & \varepsilon''_j \\ \varepsilon_1\varepsilon_2 & \varepsilon'_1\varepsilon'_2 & \varepsilon''_1\varepsilon''_2 \end{pmatrix}$$

and

$$B_\rho = \begin{pmatrix} \rho\omega_1 & \rho\omega_2 & \rho\omega_3 \\ \rho'\omega'_1 & \rho'\omega'_2 & \rho'\omega'_3 \\ \rho''\omega''_1 & \rho''\omega'''_2 & \rho''\omega''_3 \end{pmatrix}.$$

For $\tau_1, \tau_2 \in K$, $\nu = 1, 2$, set

$$M(2, \nu, \tau_1, \tau_2) := 0$$

if $\det E_\nu = 0$, *otherwise*

$$\begin{aligned} M(2, \nu, \tau_1, \tau_2) &: \\ = &-\frac{4\pi^6}{135}\operatorname{sign}(L)(-1)^\nu N(\rho)^2 \frac{\det E_\nu}{|\det(E_\nu B_\rho)|^3} \\ &\cdot \sum_{m_1=0}^{6}\sum_{m_2=0}^{6} \binom{6}{m_1,\, m_2} \\ &\cdot \{B(3, m_1, m_2, 6-(m_1+m_2), (E_\nu B_\rho)^*, 0) \\ &\cdot \sum_{\kappa_1=0}^{1}\sum_{\kappa_2=0}^{1}\sum_{\mu_1=0}^{1}\sum_{\mu_2=0}^{1} \binom{m_1 - 1}{1-(\kappa_1+\kappa_2),\, 1-(\mu_1+\mu_2)} \\ &\cdot \binom{m_2 - 1}{\kappa_1,\, \mu_1}\binom{5-(m_1+m_2)}{\kappa_2,\, \mu_2} \\ &\cdot Tr_{K/\mathbf{Q}}(\tau_1^{\kappa_1+\kappa_2}\tau_1'^{\mu_1+\mu_2}\tau_1''^{4-(m_1+\kappa_1+\kappa_2+\mu_1+\mu_2)} \\ &\cdot \tau_2^{\kappa_2}\tau_2'^{\mu_2}\tau_2''^{5-(m_1+m_2+\kappa_2+\mu_2)})\}, \end{aligned}$$

where $(E_\nu B_\rho)^*$ denotes the transposed matrix of $(E_\nu B_\rho)$, and

$$\begin{aligned} C(2, \nu, \tau_1, \tau_2) &: \\ = &-\frac{4\pi^6}{3}\operatorname{sign}(L)(-1)^{\nu+1} \\ &\cdot N(\rho)^2\tilde{B}_4(0)|\det B_\rho|^{-1}\operatorname{sign}(\det E_\nu) \\ &\cdot \{\operatorname{sign}((\tau_1\tau_2 - \tau'_1\tau'_2)(\tau_1 - \tau'_1)) \\ &\qquad + \operatorname{sign}((\tau'_1\tau'_2 - \tau''_1\tau''_2)(\tau'_1 - \tau''_1)) \\ &+ \operatorname{sign}((\tau''_1\tau''_2 - \tau_1\tau_2)(\tau''_1 - \tau_1)) \\ &\qquad + \operatorname{sign}((\tau''_1(\tau_1 - \tau'_1)(\tau'_2 - \tau_2)) \\ &+ \operatorname{sign}((\tau_1(\tau'_1 - \tau''_1)(\tau''_2 - \tau'_2)) \\ &\qquad + \operatorname{sign}((\tau'_1(\tau''_1 - \tau_1)(\tau_2 - \tau''_2)) \\ &+ N(\tau_2)[\operatorname{sign}((\tau''_1(\tau_2 - \tau'_2)(\tau_1\tau_2 - \tau'_1\tau'_2)) \\ &+ \operatorname{sign}((\tau_1(\tau'_2 - \tau''_2)(\tau'_1\tau'_2 - \tau''_1\tau''_2)) \\ &+ \operatorname{sign}((\tau'_1(\tau''_2 - \tau_2)(\tau''_1\tau''_2 - \tau_1\tau_2))]\}. \end{aligned}$$

Define

$$\begin{aligned} \zeta(2, W, \tau_1, \tau_2) := &M(2, 1, \tau_1, \tau_2) + M(2, 2, \tau_2, \tau_1) \\ &+ C(2, 1, \tau_1, \tau_2) + C(2, 2, \tau_2, \tau_1). \end{aligned}$$

Let $\zeta(s, K_0)$ be the partial zeta function of an absolute ideal class K_0 of K and $W \in K_0^{-1}$. Then we have

$$\zeta(2, K_0) = \frac{1}{2}\operatorname{Norm}(W)^2\zeta(2, W, \varepsilon_1, \varepsilon_2).$$

Remark 1. For $k, l, m \in \boldsymbol{Z}$,

$$\binom{k}{l,\, m} := \frac{k!}{l!m!(k-(l+m))!},$$

$$\text{if } k, l, m, k-(l+m) \in \boldsymbol{N} \cup \{0\},$$

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