On Terai's conjecture*)

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Abstract: Terai presented the following conjecture: If $a^2 + b^2 = c^2$ with $a > 0$, $b > 0$, $c > 0$, gcd $(a, b, c) = 1$ and a even, then the diophantine equation $x^2 + b^m = c^n$ has the only positive integral solution $(x, m, n) = (a, 2, 2)$. In this paper we prove that if (i) b is a prime power, $c \equiv 5 \pmod 8$, or (ii) $c \equiv 5 \pmod 8$ is a prime power, then Terai's conjecture holds.

1. Introduction. In 1956, Jeśmanowicz [4] conjectured that if a, b, c are Pythagorean triples, i.e. positive integers a, b, c satisfying $a^2 + b^2 = c^2$, then the Diophantine equation

$$a^x + b^y = c^z$$

has the only positive integral solution $(x, y, z) = (2, 2, 2)$. When a, b, c take some special Pythagorean triples, it was discussed by Sierpinski [14], C. Ko [5-10], J. R. Chen [2], Dem'janenko [3] and others.

In 1993, as an analogue of above conjecture, Terai [16] presented the following:

Conjecture. If $a^2 + b^2 = c^2$ with gcd $(a, b, c) = 1$ and a even, then the Diophantine equation

$$x^2 + b^m = c^n \tag{1}$$

has the only positive integral solution $(x, m, n) = (a, 2, 2)$.

Terai proved that if b and c are primes such that (i) $b^2 + 1 = 2c$, (ii) $d = 1$ or even if $b \equiv 1 \pmod 4$, where d is the order of a prime divisor of $[c]$ in the ideal class group of $\boldsymbol{Q}(\sqrt{-b})$, then the conjecture holds. Further, he proved that if $b^2 + 1 = 2c$, $b < 20$, $c < 200$, then conjecture holds. Recently, X. Chen and M. Le [11] proved that if $b \not\equiv 1 \pmod{16}$, $b^2 + 1 = 2c$, b and c are both odd primes, then the conjecture holds, and P. Yuan and J. Wang [17] proved that if $b \equiv \pm 3 \pmod 8$ is a prime, then Terai's conjecture holds.

In this paper, we consider Terai's conjecture when b or c is prime power. Then we prove the following:

Theorem 1. If b is a prime power, $c \equiv 5 \pmod 8$, then Terai's conjecture holds.

Corollary. If $2k + 1$ is a prime, $k \equiv 1$ or $2 \pmod 4$, then the Diophantine equation

$$x^2 + (2k+1)^m = (2k^2 + 2k + 1)^n$$

has the only positive integral solution $(x, m, n) = (2k^2 + 2k, 2, 2)$.

Theorem 2. If $c \equiv 5 \pmod 8$ is a prime power, then Terai's conjecture holds.

2. Some lemmas. We use the following lemmas to prove our theorems.

Lemma 1. If a, b, c are positive integers satisfying $a^2 + b^2 = c^2$, where $2 \mid a$, gcd $(a, b, c) = 1$, then

$$a = 2st,\ b = s^2 - t^2,\ c = s^2 + t^2,$$

where $s > t > 0$, gcd $(s, t) = 1$ and $s \not\equiv t \pmod 2$.

Lemma 2 (Störmer [15]). The Diophantine equation

$$x^2 + 1 = 2y^n$$

has no solutions in integers $x > 1$, $y \geq 1$ and n odd ≥ 3.

Lemma 3 (Ljunggren [12]). The Diophantine equation

$$x^2 + 1 = 2y^4$$

has the only positive integral solutions $(x, y) = (1, 1)$ and $(239, 13)$.

Lemma 4 (Cao [1]). If p is an odd prime and the Diophantine equation

$$x^p + 1 = 2y^2 (|y| > 1)$$

has integral solution x, y, then $2p \mid y$.

Now, we assume that a, b, c are Pythagorean triples with gcd $(a, b, c) = 1$ and $2 \mid a$.

Lemma 5. If $c \equiv 5 \pmod 8$, then we have

$$(b/c) = (c/b) = -1,$$

where $(*/*)$ denotes Jacobi's symbol.

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