Construction of Jacobi cusp forms

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1. Introduction. In [2 ; Theorem 3. 1], certain linear operators on spaces of Jacobi forms were introduced for cusp forms of one variable. These operators are represented as adjoint operators to "product operators" with respect to the Petersson inner product. The purpose of the present paper is to extend this result to the case of general "Jacobi cusp forms" in place of cusp forms of one variable (see Theorem 3.1 below). This extension is obtained in the same way as in [2], but the L-series which appear in our Theorem are of a different type from those in the theorem in [2].

2. Jacobi forms and Petersson inner product. For the theory of Jacobi forms we refer to [3]. We write $\Gamma_1 = SL_2(\boldsymbol{Z})$ for the full modular group and $\mathfrak{H}$ for the upper half-plane. The elements $\gamma = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu))$ of the Jacobi group $\Gamma_1^J = \Gamma_1 \ltimes \boldsymbol{Z}^2$ operates on $\mathfrak{H} \times \boldsymbol{C}$ in the usual way by

$$\gamma \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right)$$

and for given positive integers k and m on functions $\phi : \mathfrak{H} \times \boldsymbol{C} \to \boldsymbol{C}$ by

$$\phi|_{k,m}\gamma = (c\tau + d)^{-k} \times \exp(-2\pi i m(\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} - \lambda^2\tau - 2\lambda z)) \times \phi(\gamma \circ (\tau, z)).$$

Let $J_{k,m}$ be the space of Jacobi forms of weight k and index m, i.e. the space of holomorphic functions $\phi : \mathfrak{H} \times \boldsymbol{C} \to \boldsymbol{C}$ satisfying $\phi|_{k,m}\gamma = \phi$ for all $\gamma \in \Gamma_1^J$ and let the Fourier expansion of ϕ be

$$\phi(\tau, z) = \sum_{n,r \in Z, r^2 \leq 4mn} c(n, r) e^{2\pi i(n\tau + rz)}.$$

We write $J_{k,m}^{\text{cusp}}$ for the subspace of cusp forms of $J_{k,m}$, which require $c(n, r) = 0$ unless $r^2 < 4mn$. For $\phi_1, \phi_2 \in J_{k,m}$ such that $\phi_1 \times \phi_2$ is cuspidal, we define the Petersson inner product by

$$< \phi_1, \phi_2 > = \int_{\Gamma_1^J \backslash \mathfrak{H} \times C} \phi_1(\tau, z) \overline{\phi_2(\tau, z)} v^k e^{-4\pi m y^2/v} dV_J,$$

where $\tau = u + iv$, $z = x + iy$ and $dV_J = v^{-3}$ $dudvdxdy$ is an invariant measure under the action of Γ_1^J on $\mathfrak{H} \times \boldsymbol{C}$. The space $(J_{k,m}^{\text{cusp}}, <, >)$ is a finite dimensional Hilbert space. The following lemma will be used later.

Lemma 2.1. *Let ϕ be a function in $J_{k,m}$ with Fourier coefficients $c(n, r)$ and put the discriminant $D := r^2 - 4mn$. Then $c(n, r)$ depend only on D and on the residue class of r modulo $2m$. Furthermore, if $k > 3$ and ϕ is a cusp form, then*

$$c(n, r) \ll |D|^{k/2-1/2} (D < 0).$$

Remark 2.1. *If we have only the condition $k > 3$, then*

$$c(n, r) \ll |D|^{k-3/2} (D < 0).$$

For a proof of the first statement, see [3, pp. 22-23], and for the second statement (the estimates of Fourier coefficients), see [1, pp 308]. Hereafter we shall write simply $c_\mu(D')$ for $c(n, r)$ where $D' = |D| = 4mn - r^2$ and $\mu \equiv r \pmod{2m}$.

3. Construction of Jacobi forms. First we remind the definition of the Jacobi Poincaré series.

Definition 3.1. *For $n, r \in \boldsymbol{Z}$ with $r^2 < 4mn$ we denote by*

$$P_{k,m;n,r}(\tau, z) = \sum_{\gamma \in \Gamma_{1,\infty}^J \backslash \Gamma_1^J} e^{2\pi i(n\tau + rz)}|k, m\gamma$$

the (n, r)-th Jacobi Poincaré series of weight k and index m. (Note that the group

$$\Gamma_{1,\infty}^J = \left\{ \left[\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0, \mu) \right] \middle| t, \mu \in \boldsymbol{Z} \right\}$$

is a stabilizer of $e^{2\pi i(n\tau + rz)}$ in Γ_1^J.)

It is well-known that $P_{k,m;n,r}(\tau, z) \in J_{k,m}^{\text{cusp}}$ for weight $k > 2$ (see [4]). This (infinite) series has the following property, and it is expanded in a neighborhood of cusp as follows.

Lemma 3.1. *Let $\phi(\tau, z) \in J_{k,m}^{\text{cusp}}$ with $\phi(\tau, z) = \sum_{n,r \in Z, r^2 < 4mn} c(n, r) e^{2\pi i(n\tau + rz)}$. Then*

$$< \phi(\tau, z), P_{k,m;n,r}(\tau, z) > = \alpha_{k,m} (4mn - r^2)^{3/2-k} c(n, r),$$

where

$$\alpha_{k,m} = \frac{m^{k-2} \Gamma(k - 3/2)}{2\pi^{k-3/2}}.$$