Interacting Brownian motions with measurable potentials

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1. Introduction. We construct (unlabeled) interacting Brownian motions, so-called infinite dimensional Wiener processes with interaction, by using Dirichlet form theory.

(Labeled) interacting Brownian motions are infinitely dimensional diffusion processes with state space $(\boldsymbol{R}^d)^N$ given by the following SDE;

$$(1.1)\quad dX_t^i = dB_t^i - \sum_{j=1, j\neq i}^{\infty} \frac{1}{2} \nabla \Phi(X_t^i - X_t^j)dt \qquad (1 \le i < \infty),$$

where B_t^i are independent Brownian motion on $\boldsymbol{R}^d$. When $\Phi \in C_0^3(\boldsymbol{R}^d)$, this equation was solved by Lang [3], [4]. (see [1], [5], [8], [11], [12] for further development). The Θ-valued diffusion process associated with (1.1) (unlabeled interacting Brownian motion) is

$$\Xi_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \ (\delta_a \text{ is the delta measure at } a).$$

Diffusion processes $\{P_\theta\}_{\theta\in\Theta}$ obtained in Corollary 1.3 below is corresponds to Ξ_t. We refer to Theorem 3 in [7] with Remark (3,4) after that for the precise meaning of *correspondence* and related open problems.

We assume interacting potential Φ is super stable and lower regular in the sense of Ruelle, and there exists a upper semicontinuous function $\tilde{\Phi}$ that are regular in the sense of Ruelle and dominates Φ from above. We remark Φ itself is not necessarily upper semicontinuous; Φ needs no regularity more than measurability. We henceforth generalize results in [7] and [13].

Let Θ be the set of all locally finite configurations on $\boldsymbol{R}^d$. Here a configuration θ is a Radon measure of the form $\theta = \sum_i \delta_{x^i}$, where $\{x^i\}$ is a finite or infinite sequence in $\boldsymbol{R}^d$ with no cluster points. We endow Θ with the vague topology; Θ is a Polish space with this topology.

Let Φ: $\boldsymbol{R}^d \to \boldsymbol{R} \cup \{\infty\}$ be a measurable function such that $\Phi(x) = \Phi(-x)$. We assume:

(Φ.1) Φ is super stable in the sense of Ruelle. (see [9] and [10]).

(Φ.2) Φ is lower regular in the sense of Ruelle; there exist a positive, decreasing function φ: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ satisfying

$$\int_{\boldsymbol{R}^+} \varphi(t)t^{d-1}dt < \infty,$$
$$\Phi(x) \ge -\varphi(|x|) \text{ for all } x \in \boldsymbol{R}^d.$$

(Φ.3) There exists a upper semicontinuous function $\tilde{\Phi}$: $\boldsymbol{R}^d \to \boldsymbol{R} \cup \{\infty\}$ and a constant $R > 0$ such that

$$\Phi(x) \le \tilde{\Phi}(x) \text{ for all } x \in \boldsymbol{R}^d,$$
$$\tilde{\Phi}(x) \le \varphi(|x|) \text{ for all } |x| \ge R,$$
$$\tilde{\Phi}(x) = \infty \text{ if and only if } \Phi(x) = \infty.$$

Here φ is same as (Φ.2).

We remark by (Φ.1) Φ is bounded from below. By (Φ.1) − (Φ.3) for each $z > 0$ there exist (grand canonical) Gibbs measures μ with pair potential Φ and activity z ([10]). The definition of Gibbs measure will be given in Section 2.

We consider a symmetric bilinear form $\mathscr{E}$ on Θ;

$$\mathscr{E}(f, g) = \int_\Theta D[f, g]d\mu.$$

Here $D[f, g]$ is given by

$$D[f, g](\theta) = \frac{1}{2}\sum_i \nabla_i \hat{f}(x) \cdot \nabla_i \hat{g}(x).$$

Here $\nabla_i = (\frac{\partial}{\partial_{x_{ik}}})_{1 \le k \le d}$, and $\cdot$ means the inner product on $\boldsymbol{R}^d$. $\hat{f}$ and $\hat{g}$ in the right hand side are permutation invariant functions given by $f(\theta) = \hat{f}(x)$ and $g(\theta) = \hat{g}(x)$, where $x = (x^i)$ is such that $\theta = \sum_i \delta_{x^i}$. Bilinear map $D[f, g]$ is defined on $\mathscr{D}_\infty^{loc.}$, the space of local, smooth functions on Θ, defined in Section 2. Let

$$\mathscr{D} = \{f \in \mathscr{D}_\infty^{loc.}; \mathscr{E}(f, f) < \infty, \|f\|_{L^2(\Theta,\mu)} < \infty\}.$$

The purpose of this paper is to prove $(\mathscr{E}, \mathscr{D})$ is closable on $L^2(\Theta, \mu)$.

Theorem. 1.1. *Assume* (Φ.1) − (Φ.3). *Let* μ *be a Gibbs measure with potential* Φ. *Then* $(\mathscr{E}, \mathscr{D}_\infty)$ *is closable on* $L^2(\Theta, \mu)$.

Remark 1.1. In the previous work [7] we proved this result under more restrictive assumptions (Φ.1), (Φ.2) and (Φ.3′), (Φ.4′) below:

(Φ.3′) Φ is tempered in the sense of Ruelle; there exist a decreasing function $\varphi: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ and a constant R_1 such that