# Uniqueness of Unibranched Curve in $\boldsymbol{R}^2$ up to Simple Blowings up

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**Introduction.** We consider the case of two real variables. A function $g:\boldsymbol{R}^2\to\boldsymbol{R}$ is blow-analytic if there exists a composition of simple blowings up, each centered at a point, $\beta=\beta_1\circ\cdots\circ\beta_n:X\to\boldsymbol{R}^2$ such that $g\circ\beta$ is analytic ([7], [4]). If a homeomorphism $h:\boldsymbol{R}^2\to\boldsymbol{R}^2$ and its inverse, $h^{-1}$, both have blow-analytic components, we say $h$ is blow-analytic.

S. Koike ([5]) was the first to discover a blow-analytic homeomorphism of $\boldsymbol{R}^3$ with itself which is not Lipschitz (see also [4]). L. Paunescu, in [9], has discovered one, also of $\boldsymbol{R}^3$, which does not preserve the multiplicity of analytic arcs.

On the other hand, M. Suzuki ([11]) and T. Fukui ([3]) have found some blow-analytic invariants which can be used to show, for example, that functions like $x$, $x^2-y^3$, $x^3-y^7$ are not blow-analytically equivalent. A seemingly simple question, raised by Koike, is whether a blow-analytic homeomorphism $h:\boldsymbol{R}^2\to\boldsymbol{R}^2$ can carry a line $\{x=0\}$ to a singular curve such as $\{x^2=y^3\}$ or $\{x^3=y^7\}$. (These are as topological spaces; analytic structures are ignored.)

We shall answer the question in the affirmative.

Let $C$ be the germ of a singular curve in $\boldsymbol{R}^2$, which is unibranched in the sense that its complexification has only one branch ([12]).

We first take a sequence of blowings up to desingularize $C$ to a smooth curve, $\tilde{C}$, transversal to exactly one exceptional curve, we then prove that there is a way to blow down to $\boldsymbol{R}^2$ again so that the images of $\tilde{C}$, at all stages, remain smooth. This is the content of the Main Theorem (Theorem 8).

Our results and techniques are developped merely for the real analytic case. We no longer have them in the complex case, where things are too rigid.

**Notation and an invariant for real exceptional curves.** First we define an invariant for simple blowings up on surfaces.

**Proposition 1.** *There are only two real analytic equivalence classes of germs of compact connected smooth real analytic manifold of dimension one embedded in a smooth surface; that is,*

(1) *the direct product of the germ of a one dimensional open segment and* $\boldsymbol{RP}^1$ *or*

(2) *the tubular neighbourhood of the exceptional divisor of the blowing up of the origin of* $\boldsymbol{R}^2$.

*Topologically* (1) *is an annulus,* (2) *is a Möbius band.*

We say a neighbourhood of $\boldsymbol{RP}^1$ is *even* for (1) and *odd* for (2). We also call the central curve *even* and *odd* accordingly.

**Definition 2.** We call a blowing-up of a maximal ideal of a smooth point in a surface a *simple blowing up.*

The exceptional curve is an odd curve. A simple blowing up centered at a point on a curve changes the parity of the curve.

**Proposition 3.** *Any real analytic curve singularity embedded in a smooth surface can be resolved by a sequence of simple blowings up.*

Take a sequence of simple blowings-up $\beta=\beta_n\circ\cdots\circ\beta_1:X=X_n\to X_0=\boldsymbol{R}^2$, $\beta_i:X_i\to X_{i-1}$, whose centers are all above the origin of $\boldsymbol{R}^2$. The whole sequence is just the real slice of a natural composition of point blowings-up of $\boldsymbol{C}^2$. The exceptional set in $X$ is a connected union of normal crossing real projective lines $E_i$ which is the strict transform of the exceptional divisor of $\beta_i$. We associate to $(X, \{E_i\})$ a matrix whose entries are in $\boldsymbol{Z}/2\boldsymbol{Z}$.

**Definition 4.** The *intersection matrix* of $(X, \{E_i\})$ is an $n\times n$ matrix $(a_{ij})$ where

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