A Remark on the Chern Classes of Local Complete Intersections

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§0. Introduction. For a possibly singular complex algebraic or analytic variety X there are (at least) three kinds of Chern classes available. One is the *Chern-Schwartz-MacPherson class* [3 and 17], denoted $C^{SM}(X)$. This was first constructed by M.-H. Schwartz using radial vector fields, then its existence as a natural transformation of functors was conjectured by P. Deligne and A. Grothendieck and was proved by R. MacPherson. Another is the *Chern-Mather class*, denoted $C^{M}(X)$. This is defined via the Nash blow-up and is, roughly speaking, the Chern class of the limiting tangent bundle of the smooth part of X. The relation between these two classes is another aspect of MacPherson's theory, which expresses $C^{SM}(X)$ in terms of $C^{M}(M)$ and the extra terms supported on the singular locus. This theorem is proved by introducing the *local Euler obstruction,* which also appears in the Dubson-Kashiwara index [4 and 12]. The third is the *canonical class* or *Fulton-Johnson's Chern class* [7 and 8], denoted $C^{FJ}(X)$. This is defined in terms of the Segre class of X and is relatively easy to understand when X is a local complete intersection.

These three classes are identical when the variety has no singularities, thus the differences among them are expected to be expressible in terms of certain invariants of singularities. For a (strong) local complete intersection X with isolated singularities, in [19] is proved a formula expressing $C^{SM}(X)$ in terms of $C^{FJ}(X)$ and the Milnor numbers of the singularities. The purpose of this note is to report an observation that this formula together with other already known formulas implies an interesting and possibly promising formula relating $C^{M}(X)$ and $C^{FJ}(X)$ for such varieties X (see Theorem 3.3 below).

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§1. Three Chern classes. In this section we give a brief review on the above mentioned Chern classes. First, to define the Chern-Mather class, let $\nu : \hat{X} \to X$ be the Nash blow-up of X with $\widehat{TX}$ the Nash tangent bundle over $\hat{X}$. Then the Chern-Mather class is defined by

$$C^{M}(X) := \nu_*(c(\widehat{TX}) \frown [\hat{X}]),$$

where $c(\widehat{TX})$ is the usual total Chern cohomology class of the vector bundle $\widehat{TX}$ and $[\hat{X}]$ is the fundamental class of $\hat{X}$.

Using the Chern-Mather class we can define the Chern-Schwartz-MacPherson class. Let $\mathcal{F}(X)$ be the abelian group of constructible functions on X, which is freely generated by the local Euler obstruction functions Eu_W's of reduced, irreducible subvarieties W's of X. It is proved in [17] that there exists a unique natural transformation $C_* : \mathcal{F} \to H_*(\ ;\boldsymbol{Z})$ satisfying the extra condition that, if X is smooth then $C_*(1_X) = c(TX) \frown [X]$, the Poincaré dual of the total Chern cohomology class of the tangent bundle TX. In fact, C_* is given by $C_*(\sum_W Eu_W) := \sum_W C^{M}(W)$. Then the Chern-Schwartz-MacPherson class of X is defined by:

$$C^{SM}(X) := C_*(1_X).$$

Since we may write $1_X = Eu_X + \sum_S n_S Eu_S$ for uniquely determined subvarieties S of the singular locus of X and integers n_S, we have

$$C^{SM}(X) = C^{M}(X) + \sum_S n_S C^{M}(S).$$

Let X be a local complete intersection in a complex analytic manifold M. Then the normal bundle to the smooth part of X can be extended to a vector bundle N_X over the whole X. More precisely, let $\mathcal{I}_X$ be the ideal sheaf of X in the structure sheaf $\mathcal{O}_M$ of M and $\mathcal{O}_X = \mathcal{O}_M/\mathcal{I}_X$, then the vector bundle N_X is identified with the normal sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X)$, which is locally free in this case. For such X, we have the virtual tangent bundle $TM|_X - N_X$, whose total Chern cohomology class is given by $c(TM|_X - N_X) = \dfrac{c(TM|_X)}{c(N_X)}$. Then Fulton-Johnson's Chern class in