Range Theorems and Inversion Formulas for Radon Transforms on Grassmann Manifolds

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(Communicated by Kiyosi ITÔ, M. J. A., May 12, 1997)

Abstract: In this article, we state some results on the range characterization for Radon transforms on Grassmann manifolds and give the explicit inversion formulas.

Key words: Grassmann manifold; integral geometry; inversion formula; Radon transform; range theorem.

1. Introduction. Let us denote by $\boldsymbol{F}$ the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$. Let $Gr(p, n; \boldsymbol{F})$ be the Grassmann manifold of all p-dimensional subspaces in $\boldsymbol{F}^n$. Then, the Radon transform $R_p^q : C^\infty(Gr(q, n; \boldsymbol{F})) \to C^\infty(Gr(p, n; \boldsymbol{F}))$ is defined as follows.

(1.1) $$R_p^q f(\xi) := \int_{\{\gamma;\gamma\subset\xi\}} f(\gamma)\, d\mu, \quad \text{if } q < p,$$

(1.2) $$R_p^q f(\xi) := \int_{\{\gamma;\gamma\supset\xi\}} f(\gamma)\, d\mu, \quad \text{if } p < q,$$

for a p-dimensional subspace $\xi \in Gr(p, n; \boldsymbol{F})$ and for $f \in C^\infty(Gr(q, n; \boldsymbol{F}))$. Here in (1.1) or in (1.2), $d\mu$ denotes the normalized invariant measure.

Let $s := \min\{q, n-q\} = \operatorname{rank} Gr(q, n; \boldsymbol{F})$ and $r := \min\{p, n-p\} = \operatorname{rank} Gr(p, n; \boldsymbol{F})$. If $s < r (\Leftrightarrow \dim Gr(q, n; \boldsymbol{C}) < \dim Gr(p, n; \boldsymbol{F}))$, the Radon transform R_p^q is no longer surjective. On the other hand, it is known that R_p^q is injective if $s \le r$. Thus, we arrive at the problems; how to characterize the range of R_p^q and how to reconstruct the inverse image of R_p^q. In fact, for the first problem, Gonzalez [1] shows the existence of the range characterizing operator for R_p^q and for the second problem, Grinberg [3] shows the existence of the inversion formula for R_p^q. However, explicit results for these two problems are still unknown. Therefore, in this article, we give the explicit form of the range characterizing operator and the explicit inversion formula for the Radon transform R_p^q.

The integral geometry on Grassmann manifolds and related subjects will be investigated in our forthcoming paper [9], in which the results in this article will be proved.

2. Complex case. In this section, we deal with the case of complex Grassmann manifolds. The special unitary group $G := SU(n)$ acts on the complex Grassmann manifold $Gr(p, n; \boldsymbol{C})$ transitively. The stabilizer of the p-dimensional subspace $\boldsymbol{C}\boldsymbol{e}_1 \oplus \cdots \oplus \boldsymbol{C}\boldsymbol{e}_p$ is $K_p := S(U(p) \times U(n-p))$. Then, $Gr(p, n; \boldsymbol{C})$ can be identified with the compact symmetric space G/K_p.

First, we construct a certain kind of differential operators on the complex Grassmann manifold $Gr(p, n; \boldsymbol{C})$, which are expressed in terms of determinantal type of differential operators.

Let $\mathfrak{g}$ and $\mathfrak{k}_p$ denote the Lie algebras of G and of K_p, respectively. Then,

$$\mathfrak{g} = \{X \in M_n(\boldsymbol{C}) ; X + X^* = 0, \operatorname{tr} X = 0\},$$

$$\mathfrak{k}_p = \left\{\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g} ; X_1 \in M_p(\boldsymbol{C}), X_2 \in M_{n-p}(\boldsymbol{C})\right\}.$$

Let $\mathfrak{g} = \mathfrak{k}_p \oplus \mathfrak{M}_p$ be the Cartan decomposition of the symmetric space G/K_p, where $\mathfrak{M}_p$ is the space of all the matrices X of the form

(2.1) $$X = \begin{pmatrix} 0 & -Z^* \\ Z & 0 \end{pmatrix} \in \mathfrak{g},$$

$$Z = (z_{i\alpha}) : \text{complex } (n-p) \times p \text{ matrix}$$
$$(1 \le i \le n-p,\ 1 \le \alpha \le p).$$

Let$I = \{i(1), i(2), \ldots, i(d) ; 1 \le i(1) < i(2) < \cdots < i(d) \le n-p\}$ and $A = \{\alpha(1), \alpha(2), \ldots, \alpha(d), ; 1 \le \alpha(1) < \alpha(2) < \cdots < \alpha(d) \le p\}$ be two ordered sets.

For the submatrix Z of X in (2.1) and the above two ordered sets I and A, we define $d \times d$ matrix valued differential operators $\partial Z_{(I,A)}$ and $\partial \bar{Z}_{(I,A)}$ by

(2.2) $$\partial Z_{(I,A)} := \left(\frac{\partial}{\partial z_{i\alpha}}\right)_{i\in I, \alpha\in A},$$

Mathematics Subject Classification (1991): 44A12, 65R10, 92C55.