Lie Extensions

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1. Introduction. In [8], Vessiot investigated the following system of ordinary differential equations

(1) $$\frac{dy_i}{dx} = \sum_{j=1}^{m} a_j X_j y_i \quad (1 \le i \le n),$$

which he called a "Lie system" after Lie's work [3]. Here the a_j denote functions in the independent variable x and X_j are linear differential operators in the shape

$$X_j = \sum_{i=1}^{n} \xi_{ji} \frac{\partial}{\partial y_i} \ (1 \le j \le m)$$

with the ξ_{ji} being functions of y, which constitute a Lie algebra over the field of complex numbers. Consideration of integrals of this turns out to be the same as of the differential operator

$$D = \frac{\partial}{\partial x} + \sum_{j=1}^{m} a_j X_j, \quad \left[\frac{\partial}{\partial x}, X_j\right] = 0,$$

which must satisfy

$$[D, X_i] = \sum_{j=1}^{m} \sum_{k=1}^{m} a_j c_{ijk} X_k,$$

with the c_{ijk} being the structure constants of the Lie algebra.

Here we shall examine the relationship between Lie systems and strongly normal extensions. To do that some preliminaries may be needed. Let K be an ordinary differential field of characteristic 0 with the differentiation D. In what follows we assume that the field of constants C_K of K is algebraically closed. Differential field extensions of K would be referred to be finitely generated as field extensions without notice. For a differential field extension R/K we adopt the usual notation $Der(R/K)$ for the Lie algebra consisting of all derivations of R over K. Differentiation D of R can be regarded as contained in $Der(R/C_K)$. Hence we can define the Lie product $[D, X]$ for $X \in Der(R/K)$, which is seen to lie therein. Let us denote by $\Omega^1(R/K)$ the dual R-vector space of $Der(R/K)$. It is generated with the differentials da of $a \in R$. Here $da(X) = Xa$ for $X \in Der(R/K)$. An additive endomorphism D of $\Omega^1(R/K)$ is defined by

$$(D\omega)X = D(\omega X) - \omega[D, X]$$
$$(\omega \in \Omega^1(R/K),\ X \in Der(R/K)).$$

Clearly $D(adb) = D(a)db + adDb$ holds for $a, b \in R$. Denote by $G(R/K)$ the group of all differential automorphisms of R/K. For every $\sigma \in G(R/K)$ we define two additive automorphisms σ_* and σ^* of $Der(R/K)$ and $\Omega^1(R/K)$ respectively by

$$\sigma_* X = \sigma X \sigma^{-1} \ (X \in Der(R/K)),$$
$$\sigma^* \omega = \sigma \omega \sigma_*^{-1} \ (\omega \in \Omega^1(R/K)).$$

Then $\sigma^*(adb) = \sigma(a) d\sigma b$ for $a, b \in R$.

Definition 1. We say that a differential field extension R/K is a *Lie extension* if $C_R = C_K$, there exists a C_K-Lie subalgebra $\mathfrak{g}$ of $Der(R/K)$ of finite dimension over C_K such that $[D, \mathfrak{g}] \subset K\mathfrak{g}$ and $R\mathfrak{g} = Der(R/K)$. In this case $\mathfrak{g}$ will be called its structure.

For instance we shall prove the following theorem:

Theorem 1. *Suppose that K is algebraically closed. Then every intermediate differential field of a strongly normal extension of K is a Lie extension of K.*

Recall that a differential field extension N/K is said to be strongly normal if $C_N = C_K$ and for every differential isomorphism σ

$$N\sigma N = NC_{N\sigma N} = \sigma NC_{N\sigma N}$$

holds. $G(N/K)$ turns out to be an algebraic group defined over C_K with the dimension equal to t.d. N/K. The structure of Lie extension N/K ocasionally can be constructed from invariant derivations, ones exchangeable with every differential automorphisms (cf. [2]).

In fact, strongly normal extensions are seen to be Lie extensions of the following special type.

Definition 2. A differential field extension R/K with $C_R = C_K$ is said to be *Lie closed* if $\Omega^1(R/K)$ possesses a basis of differentials which are annihilated by D. As seen in later, Lie closed extensions are Lie extensions.

A differential field extension R/K is said to depend rationally on arbitrary constants if there