Twist and Generalized Chebyshev Polynomials

By Fumio HAZAMA

Department of Natural Sciences, College of Science and Engineering, Tokyo Denki University

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1. Introduction. In the article [2], we investigated a possible generalization of the Chebyshev polynomials $T_n(x)$, $U_n(x)$ $(n = 0,1, \cdots)$, focusing on the Diophantine equation satisfied by them: $(*)$ $T_n(x)^2 - (x^2 - 1)U_{n-1}(x)^2 = 1$, $n = 1,2,\cdots$. The crucial idea of [loc. cit.] was to regard this as a defining equation of a twist of a conic by itself. As a natural generalization, we considered a twist of a conic by an arbitrary hyperelliptic curve, and obtained a family of Diophantine equations which have solutions in a certain one-parameter family of polynomials in one variable. In the present article, we proceed to higher dimensional cases and consider the twist of a double cover of the affine space of dimension $N \geq 1$ by itself. As a result, we find certain families of polynomials in N variables, called the *generalized Chebyshev polynomials*, which enjoy a lot of fundamental properties similar to the ones the usual Chebyshev polynomials do. The purpose of this article is to announce these properties. Details will appear elsewhere.

2. Twist and generalized Chebyshev polynomials. Let k be an arbitrary field of characteristic $\neq 2$. Let $\boldsymbol{G}_{\boldsymbol{m}}$ denote the multiplicative group and let T_N denote the norm torus of dimension N. It is defined to be the kernel of the norm map $\boldsymbol{G}_{\boldsymbol{m}}^{N+1} \to \boldsymbol{G}_{\boldsymbol{m}}$ given by the formula: $(u_1, \cdots, u_{N+1}) \mapsto \prod_{1 \leq i \leq N+1} u_i$. The norm torus T_N is stable under the natural action of the symmetric group S_{N+1} of degree $N+1$ on $\boldsymbol{G}_{\boldsymbol{m}}^{N+1}$. Hence, if we denote by A_{N+1} the alternating group of degree $N + 1$, then we have quotient maps: $T_N \xrightarrow{p} T_N/A_{N+1} \xrightarrow{q} T_N/S_{N+1}$. We denote by $\Delta(u_1, \cdots, u_{N+1})$ the difference product $\prod_{1 \leq i<j \leq N+1} (u_i - u_j)$, and by $D = D(x_1, \cdots, x_N)$ its square: $D = D(x_1, \cdots, x_N) = (\Delta(u_1, \cdots, u_{N+1}))^2$. Then the quotient T_N/A_{N+1} is defined by the equation $y^2 = D(x_1, \cdots, x_N)$, where $x_k (1 \leq k \leq N)$ denote the k-th elementary symmetric polynomial. The rational maps p, q are given by the formulas:

$$p(u_1, \cdots, u_{N+1}) = (x_1, \cdots, x_N, \Delta),$$
$$q(x_1, \cdots, x_N, y) = (x_1, \cdots, x_N).$$

The n-th power endomorphism of $\boldsymbol{G}_{\boldsymbol{m}}^{N+1}$ induces the endomorphism $[n]$ of T_N, and it commutes with the action of S_{N+1}. Therefore we have the following commutative diagram:

$$\begin{array}{ccc} T_N & \xrightarrow{[n]} & T_N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ T_N/A_{N+1} & \xrightarrow{[n]} & T_N/A_{N+1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ T_N/S_{N+1} & \xrightarrow{[n]} & T_N/S_{N+1}. \end{array}$$

(Here we used the same symbol $[n]$ for the induced maps). Let T_N' denote the twist of T_N/A_{N+1} by the quadratic extension $k(T_N/A_{N+1})/k(T_N/S_{N+1})$, where $k(X)$ denotes the rational function field of a variety X defined over k. The twist T_N' is defined over $k(T_N/S_{N+1}) \cong k(x_1, \cdots, x_N)$ and its defining equation is given by the following:

$$T_N' : D(x_1, \cdots, x_N)Y^2 = D(x_1, \cdots, X_N),$$

where the capital letters $X_1, \cdots, X_N$, Y are regarded as variables (see [2] for the fundamental properties of twists). As for the set $T_N'(k(x_1, \cdots, x_N))$ of $k(x_1, \cdots, x_N)$-rational points of T_N', we have the following theorem which can be proved in the same way as in [2]:

Theorem 2.1. *There is a natural bijection between the set $T_N'(k(x_1, \cdots, x_N))$ and the set $A = \{f \in \mathrm{Rat}_k(T_N/A_{N+1}, T_N/A_{N+1}) ; f \circ \iota = \iota \circ f\}$, where $\mathrm{Rat}_k(V, W)$ for k-varieties V, W denotes the set of k-rational map of V to W, and ι denotes the involution of T_N/A_{N+1} defined by the formula $\iota(x_1, \cdots, x_N, y) = (x_1, \cdots, x_N, -y)$.*

By this theorem, the n-th power map $[n]$ corresponds to a $k(x_1, \cdots, x_N)$-rational point on the twist T_N', which we denote by

$$(t_n^{(1)}(x_1, \cdots, x_N), \cdots, t_n^{(N)}(x_1, \cdots, x_N), s_n(x_1, \cdots, x_N)).$$

We call $t_n^{(k)}(x_1, \cdots, x_N)(k = 1, \cdots, N)$ the *generalized Chebyshev polynomial of the first kind*, and $s_n(x, \cdots, x_N)$ *the generalized Chebyshev polynomial of the second kind*, because of the following natural generalization of $(*)$: