Maximal Unramified Extensions of Imaginary Quadratic Number Fields of Small Conductors

By Ken YAMAMURA

Department of Mathematics, National Defence Academy

(Communicated by Shokichi IYANAGA, M. J. A., April 14, 1997)

Let K be an algebraic number field (of finite degree) and K_{ur} its maximal unramified extension. Then the Galois group $\mathrm{Gal}(K_{ur}/K)$ can be both finite and infinite and in general it is quite difficult to determine the structure of this group. If K has sufficiently small root discriminant, then $K_{ur} = K$, that is , K has no nontrivial unramified extension. This is the case, for example, for the imaginary quadratic number fields with class number one, the cyclotomic number fields with class number one, and the real abelian number fields of prime power conductors $\leqq 67$ (see [20, Appendix]). For some fields K with small root discriminant, we can determine $\mathrm{Gal}(K_{ur}/K)$. The purpose of this article is to report that we have determined the structure of $\mathrm{Gal}(K_{ur}/K)$ of imaginary quadratic number fields K of small conductors. (Details will apear elsewhere [21]). For imaginary quadratic number fields K of conductors $\leqq 420$ ($\leqq 719$ under the Generalized Riemann Hypothesis (GRH)) we determine $\mathrm{Gal}(K_{ur}/K)$ and tabulate them for K with $K_{ur} \neq K_1$, where K_1 denotes the Hilbert class field of K. (If $K_{ur} = K_1$, then $\mathrm{Gal}(K_{ur}/K) = \mathrm{Gal}(K_1/K) \cong \mathrm{Cl}(K)$, the class group of K by class field theory). For all such K, $K_{ur} = K$, K_1, K_2, or K_3, where K_2 (resp. K_3) is the second (resp. third) Hilbert class field of K. In other words, K_{ur} coincides with the top of the class field tower of K and the length of the tower is at most three. If possible, we give also simple expressions of K_1 and K_2. Also for $K = \boldsymbol{Q}(\sqrt{d})$ with $723 \leqq |d| < 1000$, we determine $\mathrm{Gal}(K_{ur}/K)$ except for some d. (For table for such fields, see [21]).

Let $K = \boldsymbol{Q}(\sqrt{d})$ be an imaginary quadratic number field with discriminant $d < 0$. J. Martinet stated in [12] that if $|d| < 250$, then $K_{ur} = K_1$ except for 7 fields, for which he gave the structure of $\mathrm{Gal}(K_{ur}/K)$. (We note that $\mathrm{Gal}(K_{ur}/K) \cong H_{24}$ for $K = \boldsymbol{Q}(\sqrt{-248})$ in [12] is false). He also stated that this fact is proved by using the methods which J. Masely [13] (and later F. J. van der Linden [18]) used for calculation of class numbers of real abelian number fields of small conductors. They used Odlyzko's discriminant bounds and information on the structure of class groups obtained by using the action of Galois groups on class groups. In addition to their methods, we use computer for calculation of class numbers of fields of low degrees (we use KANT) and then use class number relations to get class numbers of fields of higher degrees. Results on class field towers [2, 8, 10, 11, and 17] and the knowledge of the 2-groups of orders $\leqq 2^6$ [5] and linear groups over finite fields are also used.

We know that if $|d| \leqq 499$ ($|d| \leqq 2003$ under GRH), then the degree $[K_{ur}:K]$ is finite (see [12]). For these d, we want to determine $\mathrm{Gal}(K_{ur}/K)$. The key fact is that any unramified (finite) extension L of K has the same root discriminant as $K: rd_L = |d_L|^{1/[L:\boldsymbol{Q}]} = rd_K = \sqrt{|d|}$. Thus, if we have $rd_K < B(2N)$, where $B(2N)$ denotes the lower bound for the root discriminants of the totally imaginary number fields of (finite) degrees $\geqq 2N$, then we get $[K_{ur}:K] < N$. We do not know the real values of $B(2N)$ (except for $N \leqq 4$), however, some lower bounds for $B(2N)$ are known. The best known unconditional lower bounds for $B(2N)$ can be found in the tables due to F. Diaz y Diaz [4]. If we assume the truth of GRH, much better lower bounds can be obtained. The best known conditional (GRH) lower bounds are found in the unpublished tables due to A. M. Odlyzko [14], which are copied in Martinet's expository paper [12]. Let K_l be the top of the class field tower of $K: K = K_0 \subseteqq K_1 \subseteqq K_2 \subseteqq \cdots$ (K_{i+1} is the Hilbert class field of K_i), that is, l is the smallest number with $K_{l+1} = K_l$. If we cannot get $[K_{ur}:K_l] < 60$, which implies $K_{ur} = K_l$, from available lower bounds for $B(2N)$, we need to judge whether K_l has an unramified nonsolvable Galois extension and this is