Odd Dimensional Tori and Contact Structure

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1. We investigate the problem whether or not the odd dimensional torus $T^{2n+1}(2n+1>3)$ admits a contact structure. This problem was posed on the five dimensional torus T^5 by D.E. Blair in his lecture note [1] and he showed in [1] that no torus T^{2n+1} can carry a regular contact structure.

We remark that T^3 carries a contact structure $\eta_o = \cos z dx + \sin z dy$ which is neither regular nor K-contact.

In this note we will exhibit two theorems related to the problem, of which the first one is concerned with non-existence of a K-contact structure and the second with a certain kind of one-forms not yielding a contact structure on T^5. More precisely, the theorems which we will give are the following

Theorem 1. No torus T^{2n+1} can carry a K-contact structure

and

Theorem 2. Denote by (x, y, z, u, v) the canonical coordinate in T^5. Then any one-form η of the form

$$\eta = \cos z\, dx + \sin z\, dy + f\, du + h dv$$

can not be a contact structure, provided that the functions $f, h \in C^\infty(T^5)$ satisfy either

$$\frac{\partial f}{\partial y} = \frac{\partial h}{\partial x} = 0$$

or

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial y} = 0.$$

We briefly recall the notion of contact structure by following [1].

A $(2n+1)$-dimensional manifold M is called a contact manifold when M admits a one-form η, called a contact form or a contact structure, for which the $(2n+1)$-form $\eta \wedge (d\eta)^n$ gives a volume form on M.

We call a contact structure η regular when the characteristic vector field ξ is regular.

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When a manifold M admits a contact structure η and also a metric g and a (1,1)-tensor ϕ for which the following are satisfied, (η, ϕ, g) is called a contact metric structure;

$$\phi(\phi(X)) = -X + \eta(X)\xi,$$
$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$
$$d\eta(X, Y) = 2g(X, \phi(Y)).$$

A contact metric structure (η, ϕ, g) is called K-contact when ξ is a Killing field with respect to g.

2. K-contact structures. Theorem 1 is an immediate consequence of the following theorem, since T^{2n+1} satisfies the cohomology condition in it.

Theorem 3. Let M be a $(2n+1)$-dimensional compact connected manifold. If the cohomology ring $H^*(M, \boldsymbol{R})$ satisfies

$$\wedge^{2n+1}(H^1(M, \boldsymbol{R})) = H^{2n+1}(M, \boldsymbol{R}),$$

then M can not admit a K-contact structure.

Theorem 3 is shown by applying Tachibana's theorem ([1] and [2]). In fact we have

Proof. Tachibana's theorem asserts that every harmonic one-form ω on a compact K-contact manifold satisfies $\omega(\xi) = 0$.

Suppose M admits a K-contact structure (η, ξ, ϕ, g). Then, from the cohomological assumption the $(2n+1)$-cohomology class $[dvol]$ represented by the volume form $dvol$ is given by a linear combination of $(2n+1)$-cohomology classes $[\Omega_j]$ which are represented by $(2n+1)$-exterior products of harmonic one-forms. Let $\Omega_j = \omega_1 \wedge \cdots \wedge \omega_{2n+1}$ be such a $(2n+1)$-form.

With respect to an orthonormal basis $\{e_1 = \xi, e_2, e_3, \cdots, e_{2n+1}\}$ the value $(\omega_1 \wedge \cdots \wedge \omega_{2n+1})(e_1, e_2, \cdots, e_{2n+1}) = \det(\omega_i(e_j))$ vanishes because $\omega_i(e_1) = \omega_i(\xi) = 0$ for all i. This shows that $[dvol] = 0$, yielding a contradiction. So we have Theorem 3.

Remark. Examples of manifold satisfying the cohomological condition in Theorem 3 other that T^{2n+1} are given, for instance, by the product manifolds $M = \Sigma_{g_1} \times \cdots \times \Sigma_{g_\ell} \times S^1$, $g_i > 1$