Circular Geodesic Submanifolds with Parallel Mean Curvature Vector in a Non-flat Complex Space Form

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1. Introduction. Let $f: M \to \tilde{M}$ be an isometric immersion of a connected complete Riemannian manifold M into a Riemannian manifold $\tilde{M}$. We call M a *circular geodesic* submanifold of $\tilde{M}$ provided that for every geodesic γ of M the curve $f(\gamma)$ is a circle in $\tilde{M}$. It is well-known that a round sphere is the only circular geodesic surface in $\boldsymbol{R}^3$. This result is generalized as follows: M^n is a circular geodesic submanifold of a real space form $\tilde{M}^{n+p}(c)$ of curvature c (that is, $\tilde{M}^{n+p}(c) = \boldsymbol{R}^{n+p}$, $S^{n+p}(c)$ or $\boldsymbol{RH}^{n+p}(c)$) if and only if M^n is totally umbilic in $\tilde{M}^{n+p}(c)$ or M^n is locally congruent to one of the compact symmetric spaces of rank one which is immersed into $\tilde{M}^{n+p}(c)$ with parallel second fundamental form (see, [8]).

In this paper, we consider the classification problem of circular geodesic submanifolds in a complex space form $\tilde{M}^N(c)$ of constant holomorphic sectional curvature c (that is, $\tilde{M}^N(c) = \boldsymbol{C}^N$, $\boldsymbol{CP}^N(c)$ or $\boldsymbol{CH}^N(c)$). The classification problem of circular geodesic submanifolds in a non-flat complex space form $\tilde{M}^N(c)$ is still open. In a complex space form $\tilde{M}^N(c)$, all examples of circular geodesic submanifolds what we know are parallel submanifolds (for details, see [4]). Needless to say, a parallel submanifold is not necessarily circular geodesic. The classification problem of parallel submanifolds in a non-flat complex space form was solved by Nakagawa, Naitoh, and Takagi ([5] and [6]).

Along this context, it is natural to consider the problem "In a complex space form $\tilde{M}^N(c)$ $(c \neq 0)$, does a circular geodesic submanifold have parallel second fundamental form?". We here give an affirmative partial answer to this problem. The main purpose of this paper is to prove the following.

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Theorem. *Let M be a circular geodesic submanifold of a non-flat complex space form $\tilde{M}^N(c)$. Suppose that the mean curvature vector of M is parallel with respect to the normal connection on the normal bundle $T^{\perp}M$ of M in $\tilde{M}^N(c)$. Then M has parallel second fundamental form in $\tilde{M}^N(c)$.*

2. Preliminaries. First we recall the notion of circles in a Riemannian manifold $\tilde{M}$. A curve $\gamma(s)$ of $\tilde{M}$ parametrized by arclength s is called a *circle* of curvature k, if there exists a field of unit vectors Y_s along the curve γ which satisfies the differential equations: $\nabla_{\dot{\gamma}}\dot{\gamma} = kY_s$ and $\nabla_{\dot{\gamma}}Y_s = -k\dot{\gamma}$, where k is a positive constant and $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation ∇ along γ. Next we review the notion of CR-submanifolds of a Kaehler manifold. A Riemannian submanifold M of a Kaehler manifold $\tilde{M}$ with complex structure J is called a *CR-submanifold* if there exists on M a C^∞-holomorphic distribution $\mathfrak{D}$ satisfying its orthogonal complement $\mathfrak{D}^\perp$ is a totally real distribution, i.e., $J\mathfrak{D}_p^\perp \subseteq T_p^\perp(M)$ for any $p \in M$. We note that all holomorphic submanifolds, totally real submanifolds and real hypersurfaces are necessarily CR-submanifolds. The manifold M is said to be a *λ-isotropic* submanifold of $\tilde{M}$ provided that $\|\sigma(X, X)\|$ is equal to a constant $(= \lambda)$ for all unit tangent vectors X at its each point, where σ is the second fundamental form of M in $\tilde{M}$ ([7]). In particular, the function λ is constant on M, the immersion is said to be $(\lambda-)$ *constant isotropic*. The notion of isotropic is a generalization of "totally umbilic". We remark that these two definitions are coincidental, when *codim* $M = 1$.

We now prepare the following three lemmas without proof in order to prove our Theorem:

Lemma 1 ([2]). *Let M be a Riemannian submanifold of $\tilde{M}$. Then the following two conditions are equivalent:*

(i) *M is a circular geodesic submanifold of $\tilde{M}$.*

(ii) *The submanifold M is nonzero constant isotropic and the second fundamental form σ of M in $\tilde{M}$ is*