Extensions of Hölder-McCarthy and Kantorovich Inequalities and Their Applications*)

By Takayuki FURUTA

Department of Applied Mathematics, Science University of Tokyo

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Abstract: Extensions of Hölder-McCarthy and Kantorovich inequalities are given and their applications to the order preserving power inequalities are also given.

§1. Extensions of Hölder-McCarthy and Kantorovich inequalities. This paper is an early announcement of [3], [4], and [5]. An operator means a bounded linear operator on a Hilbert space H. The celebrated Kantorovich inequality asserts that if A is positive operator on H such that $M \geq A \geq m > 0$, then $(A^{-1}x, x)(Ax, x) \leq \frac{(m+M)^2}{4mM}$ holds for every unit vector x in H. At first we state extensions of Kantorovich inequality.

Multiple positive definite operator case.

Theorem 1.1 [4]. *Let A_j be positive operator on a Hilbert space H satisfying $MI \geq A_j \geq mI (j = 1,2,\dots,k)$, where $M > m > 0$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$ and also let $x_1, x_2, \dots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then the following inequality holds;*

$$\sum_{j=1}^{k} (f(A_j)x_j, x_j) \leq \frac{(mf(M) - Mf(m))}{(q-1)(M-m)} \left(\frac{(q-1)(f(M)-f(m))}{q(mf(M)-Mf(m))}\right)^q \left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)^q$$

under any one of the following conditions (i) *and* (ii) *respectively;*

(i) $f(M) > f(m)$, $\frac{f(M)}{M} > \frac{f(m)}{m}$ *and*

$$\frac{f(m)}{m} q \leq \frac{f(M)-f(m)}{M-m} \leq \frac{f(M)}{M} q$$

holds for any real number $q > 1$,

(ii) $f(M) < f(m)$, $\frac{f(M)}{M} < \frac{f(m)}{m}$ *and*

$$\frac{f(m)}{m} q \leq \frac{f(M)-f(m)}{M-m} \leq \frac{f(M)}{M} q$$

holds for any real number $q < 0$.

*) Dedicated to Professor Shigeru Kita on his 88th birthday with respect and affection.

Corollary 1.2 [4]. *Let A_j be positive operator on a Hilbert space H satisfying $MI \geq A_j \geq mI (j = 1,2,\dots,k)$, where $M > m > 0$. Let $x_1, x_2, \dots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then the following inequality holds;*

$$\sum_{j=1}^{k} (A_j^p x_j, x_j) \leq \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)}\right)^q \left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)^q$$

under any one of the following conditions (i) *and* (ii) *respectively;*

(i) $m^{p-1} q \leq \frac{M^p - m^p}{M-m} \leq M^{p-1} q$ *holds for any real numbers $p > 1$ and $q > 1$,*

(ii) $m^{p-1} q \leq \frac{M^p - m^p}{M-m} \leq M^{p-1} q$ *holds for any real numbers $p < 0$ and $q < 0$.*

Corollary 1.2 becomes the following Corollary 1.3 if we put $q = p$.

Corollary 1.3 [4]. *Let A_j be positive operator on a Hilbert space H satisfying $MI \geq A_j \geq mI (j = 1,2,\dots,k)$, where $M > m > 0$. Let $x_1, x_2, \dots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then the following inequality holds for any real number $p \notin [0, 1]$;*

$$\sum_{j=1}^{k} (A_j^p x_j, x_j) \leq \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)}\right)^p \left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)^p.$$

Corollary 1.3 can be considered as an extension of the following Theorem A by Ky Fan.

Theorem A [1] (Ky Fan). *Let A be a positive definite Hermitian matrix of order n with all its eigenvalues contained in the closed interval $[m, M]$, where $M > m > 0$. Let $x_1, x_2, \dots, x_k$ be any finite*