# The Dynamics of Nearly Abelian Polynomial Semigroups at Infinity

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**Abstract**: We prove that a nearly abelian polynomial semigroup has the simultaneously normalizing coordinate in the neighborhood of infinity. This result has been expected by A. Hinkkanen and G. J. Martin as Conjecture 7.1 in [5].

We begin this paper with some definitions, which is given by [5].

**Definition.** A *polynomial semigroup* $G$ is a semigroup generated by a family of non-constant polynomial functions on $\boldsymbol{C} \cup \{\infty\}$ to itself. Here the semigroup operation is functional composition. And let $\langle f_1, \ldots, f_n, \ldots \rangle$ denote the semigroup generated by $f_1, \ldots, f_n, \ldots$. If $G$ is a polynomial semigroup, we define the set of normality $N(G)$ as following;

$N(G) = \{z \in \boldsymbol{C} \cup \{\infty\}$ : there is a neighborhood $V$ of $z$ such that $G|_V$ is a normal family with respect to the spherical metric$\}$.

**Definition.** A polynomial semigroup is *nearly abelian* if there is a compact family $\Phi$ of Möbius transformations with the following properties;

(i) $\phi(N(G)) = N(G)$ for all $\phi \in \Phi$, and

(ii) for all $f, g \in G$, there is a $\phi \in \Phi$ such that $f \circ g = \phi \circ g \circ f$.

Next we state our main theorem.

**Theorem 1.** *If $G$ is a nearly abelian polynomial semigroup and $G$ contains some polynomials of degree at least two, then there is a neighborhood of $\infty$ on which $G$ is analytically conjugate to a subsemigroup of $\langle z \mapsto az^n : |a| = 1, n = 1,2,3,\ldots \rangle$.*

**Remark.** The condition that "$G$ contains some polynomials of degree at least two" cannot be removed. In fact, there are counterexamples to the assertion without it. A simple example is $\langle z \mapsto 2z \rangle$.

We need two lemmas to prove Theorem 1. The first one is a consequence of Theorem 4.1 in [5].

**Lemma 2.** *Let $G$ be a nearly abelian polynomial semigroup. Then for each $g \in G$ of degree at least two, we have $N(G) = N(\langle g \rangle)$.*

The next lemma is connected with the Böttcher function. (see [3] for the proof).

**Lemma 3.** *Suppose that $f$ is a polynomial of degree $n$ which is at least two. Then there exist a neighborhood $V$ of $\infty$ and an injective holomorphic map $\varphi : V \mapsto \boldsymbol{C} \cup \{\infty\}$ such that*

(*i*) $\varphi(\infty) = \infty$,

(*ii*) $\displaystyle\lim_{z\to\infty} \frac{\varphi(x)}{z} = 1$,

(*iii*) $\varphi \circ f \circ \varphi^{-1}(\zeta) = a\zeta^n$, *where* $\zeta \in \varphi(V)$ *and* $a = \displaystyle\lim_{z\to\infty} \frac{f(z)}{z^n}$, *and*

(*iv*) *if $\Omega$ is the connected component of $N(\langle f \rangle)$ including $\infty$, then the map $z \mapsto \log|\varphi(z)|$ coincides with the Green function of $\Omega$ having the pole at $\infty$.*

**Proof of Theorem 1.** Let $g$ be an element of $G$ with degree $n$ which is at least two. Then there is a Möbius transformation $\tau$ with the property that

$$\lim_{z\to\infty} \frac{\tau \circ g \circ \tau^{-1}(z)}{z^n} = 1.$$

It is sufficient to prove this theorem that we prove the similar assertion to $\tau \circ G \circ \tau^{-1}$. Therefore we may suppose that there is a $g \in G$ such that

$$\lim_{z\to\infty} \frac{g(z)}{z^n} = 1.$$

Using Lemma 3, we obtain a neighborhood $V$ of $\infty$ and injective holomorphic map $\varphi_g : V \to \boldsymbol{C} \cup \{\infty\}$ such that

$$\varphi_g(\infty) = \infty,$$

$$\lim_{z\to\infty} \frac{\varphi_g(z)}{z} = 1,$$

and

$$\varphi_g \circ g \circ \varphi_g^{-1}(\zeta) = \zeta^n,$$

where $\zeta \in \varphi_g(V)$.