# Convergence in the Space of Fourier Hyperfunctions

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**Abstract:** A structural characterization of a convergent family of Fourier hyperfunctions $\{f_h ; h \in \Gamma\}$ is given.

**1. Notations and definitions.** We denote by $D^n$ the compactification of $R^n$, $D^n = R^n \cup S_\infty^{n-1}$ and supply it with the usual topology. The sheaves $\tilde{\mathcal{O}}$ and $\mathcal{Q}$ on $D^n + iR^n$ are defined as follows (cf. [3-6]). For any open set $U \subset D^n + iR^n$, $\tilde{\mathcal{O}}(U)$ consists of those elements of $\mathcal{O}(U \cap C^n)$ which satisfy $|F(z)| \leq C_{V,\varepsilon} \exp(\varepsilon |\mathrm{Re} z|)$ uniformly for any open set $V \subset C^n$, $\bar{V} \subset U$, and for every $\varepsilon > 0$. Hence, $\tilde{\mathcal{O}}|_{C^n} = \mathcal{O}$. The derived sheaf $\mathcal{H}^n_{D^n}(\tilde{\mathcal{O}})$, denoted by $\mathcal{Q}$, is called the sheaf of Fourier hyperfunctions. It is a flabby sheaf on $D^n$ ([4]).

Let $I$ be a convex neighbourhood of $0 \in R^n$ and $U_j = \{(D^n + iI) \cap \{\mathrm{Im} z_j \neq 0\}\}, j = 1, \dots, n$. The family $\{D^n + iI, U_j ; j = 1, \dots, n\}$ gives a relative Leray covering for the pair $\{D^n + iI, (D^n + iI) \setminus D^n\}$ relative to the sheaf $\tilde{\mathcal{O}}$. Thus $\mathcal{Q}(D^n) = \tilde{\mathcal{O}}((D^n + iI) \# D^n) / \sum_{j=1}^n \tilde{\mathcal{O}}((D^n + iI) \#_j D^n)$, where $(D^n + iI) \# D^n = U_1 \cap \dots \cap U_n$ and $(D^n + iI) \#_j D^n = U_1 \cap \dots \cap U_{j-1} \cap U_{j+1} \cap \dots \cap U_n$.

We shall use the notation $\Lambda$ for the set of $n$-vectors with entry $\{-1,1\}$; the corresponding open orthants in $R^n$ will be denoted by $\Gamma_\sigma$, $\sigma \in \Lambda$.

A global section $f = [F] \in \mathcal{Q}(D^n)$ is defined by $F \in \tilde{\mathcal{O}}((D^n + iI) \# D^n)$ ; $F = (F_\sigma)$, where $F_\sigma \in \tilde{\mathcal{O}}(D^n + iI_\sigma)$, $D^n + iI_\sigma$ is an infinitesimal wedge of type $R^n + i\Gamma_\sigma 0$, $\sigma \in \Lambda$.

Recall the topological structure of $\mathcal{Q}(D^n)$. Let $f = [F] \in \mathcal{Q}(D^n)$, $F \in \tilde{\mathcal{O}}(D^n + iI) \# D^n)$. Then, by $P_{K,\varepsilon}(F) = \sup_{z \in R^n + iK} |F(z) \exp(-\varepsilon |\mathrm{Re} z|)|$, $\varepsilon > 0$, $K \Subset I \setminus \{0\}$, is defined the family of semi-norms; $\tilde{\mathcal{O}}((D^n + iI) \# D^n)$ is a Fréchet and Montel space, as well as $\mathcal{Q}(D^n)$.

Let $f = [F] \in \mathcal{Q}(D^n)$. Then we associate to $f$, $f(x) \cong \sum_{\sigma \in \Lambda} sgn\sigma\, F_\sigma(x + i\Gamma_\sigma 0)$, $F_\sigma \in \tilde{\mathcal{O}}(D^n + iI_\sigma)$ (cf. [3], Theorem 8.5.3 and Definition 8.3.1).

The Fourier transform on $\mathcal{Q}(D^n)$ is defined by the use of functions $\chi_\sigma = \chi_{\sigma_1} \dots \chi_{\sigma_n}$, where $\sigma_k = \pm 1$, $k = 1, \dots, n$, $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\chi_1(t) = e^t/(1 + e^t)$, $\chi_{-1}(t) = 1/(1 + e^t)$, $t \in R$. Let $u(x) \cong \sum_{\sigma \in \Lambda} U_\sigma(x + i\Gamma_\sigma 0) = \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} (\chi_{\tilde{\sigma}} U_\sigma)(x + i\Gamma_\sigma 0)$, where $\chi_{\tilde{\sigma}} U_\sigma \in \mathcal{O}(D^n + iI_\sigma)$, $\sigma, \tilde{\sigma} \in \Lambda$ and decreases exponentially along the real axis outside the closed $\tilde{\sigma}$-th orthant.

The Fourier transform of $u$ is defined by

$$\mathcal{F}(u) \cong \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \mathcal{F}(\chi_{\tilde{\sigma}} U_\sigma)(x - i\Gamma_{\tilde{\sigma}} 0)$$

$$= \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \int_{\mathrm{Im} z = y^k} e^{-iz\zeta} (\chi_{\tilde{\sigma}} U_\sigma)(z) dx, \; y^k \in I_\sigma,$$

where $\mathcal{F}(\chi_{\tilde{\sigma}} U_\sigma) \in \tilde{\mathcal{O}}(D^n - iI_{\tilde{\sigma}})$ and $\mathcal{F}(\chi_{\tilde{\sigma}} U_\sigma)$ decreases exponentially along the real axis outside the closed $\sigma$-orthant.

An infinite-order differential operator $J(D) = \sum_{|\alpha| \geq 0} b_\alpha D^\alpha$ with $\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|b_\alpha| \alpha!} = 0$ is called a local operator.

**2. Convergence in $\mathcal{Q}(D^n)$.** Let $E$ be a Fréchet space with an increasing family of seminorms $\{P_i ; i \in N\}$ and let $F$ be a closed subspace of $E$. Denote by $\tilde{x}$ an element of the quotient space $E/F$ defined by $x \in E$ ; seminorms which induce the topology in $E/F$ are given by $p_i(\tilde{x}) = \inf_{y \in F} P_i(x + y)$, $i \in N$. In the sequel $\Gamma$ will be a convex cone in $R^n$.

**Proposition 1.** *A necessary and sufficient condition that a family $\{\tilde{x}_h ; h \in \Gamma\}$ converges to $\tilde{x}$ in $E/F$ as $\|h\| \to \infty$, $h \in \Gamma$, is the existence of a family $\{u_h \in E ; h \in \Gamma\}$ such that $u_h$ belongs to the class $\tilde{x}_h$ for every $h \in \Gamma$ and $u_h$ converges to $u$ in $E$ as $\|h\| \to \infty$, $h \in \Gamma$, where $u$ belongs to the class $\tilde{x}$.*

*Proof.* The sufficiency is trivial. Suppose that $\tilde{x}_h$ converges to $\tilde{x}$ in $E/F$ as $\|h\| \to \infty$, $h \in \Gamma$. Then for every $m \in N$ there exists $t_m > 0$ such that $p_m(\tilde{x}_h - \tilde{x}) = \inf_{y \in F} P_m(x_h - x + y) < 1/m$, $\|h\| \geq t_m$, $h \in \Gamma$ ; $\{t_m ; m \in N\}$ is a monotone increasing sequence which tends to infinity as $m \to \infty$. We construct a looked-for

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