On Ranks of the Stable Derivation Algebra and Deligne's Problem

By Hiroshi TSUNOGAI

Department of Mathematics, Sophia University

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1. Introduction. Among the outer Galois representations attached to algebraic varieties, the one attached to $X = \boldsymbol{P}^1 \setminus \{0,1,\infty\}$ is most fundamental. In this article, we consider the Lie algebraization of the pro-l outer Galois representation. The descending central filtration of the fundamental group $\pi_1(\bar{X})$ induces a central filtration into the absolute Galois group $G_{\boldsymbol{Q}} = \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$, and the associating graded module naturally turns into a Lie algebra, which we call the *Galois Lie algebra* in this article. In [6], Ihara introduced the *stable derivation algebra*, which includes the Galois Lie algebra. But we do not know its presentation by generators and relations, nor any explicit formulae for the ranks of its graded components. On the other hand, relating to the philosophy that not only cohomology groups but also fundamental groups are motivic [2], Deligne proposed a conjecture on the structure of the Galois Lie algebra, which is sometimes called *Deligne's motivic conjecture*. We report here that this conjecture is valid in degrees less than 13. We also obtain a partial evidence for the conjecture in degrees less than 18.

2. The Galois Lie algebra. Let l be a prime number and consider the pro-l *outer Galois representation* $\varphi_X^{(l)}$ attached to $X = \boldsymbol{P}^1 \setminus \{0,1,\infty\}$

(2.1) $$\varphi_X^{(l)} : G_{\boldsymbol{Q}} \to \mathrm{Out}\pi_1^{(l)}(\bar{X}),$$

where $G_{\boldsymbol{Q}}$ denote the absolute Galois group $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ of $\boldsymbol{Q}$, and $\pi_1^{(l)}(\bar{X})$ is the pro-l fundamental group of $\bar{X} = X \times \bar{\boldsymbol{Q}}$, which is isomorphic to the free pro-l group of rank two. By introducing the descending central filtration into $\pi_1^{(l)}(\bar{X})$, we have an injective homomorphism $\varphi_{\mathscr{G}}$ of graded Lie algebras

(2.2) $$\varphi_{\mathscr{G}} : \mathscr{G} \to \mathrm{Out}\mathscr{F}_2,$$

where $\mathrm{Out}\mathscr{F}_2$ denotes the outer derivation algebra of the free Lie algebra $\mathscr{F}_2$ of rank two [5, 8, and 9]. We call $\varphi_{\mathscr{G}}$ the *Lie algebraization* of φ_X. We also have a lifting of $\varphi_{\mathscr{G}}$ into $\mathrm{Der}\mathscr{F}_2$ and the image of $\mathscr{G}$ is included in a subalgebra $\mathscr{D}_5$ of $\mathrm{Der}\mathscr{F}_2$, whose definition we shall state later. For odd m greater than or equal to 3, the component $\mathrm{gr}^m\mathscr{G}$ of $\mathscr{G}$ of degree m has a non-trivial element σ_m which does not belong to $[\mathscr{G}, \mathscr{G}]$. This fact is deduced from the explicit formula of Ihara's power series representation and the non-trivialitity of Soulé's character. These σ_m's are called *Soulé elements*. Note that for m as above, σ_m is unique up to scalar multiple and modulo $[\mathscr{G}, \mathscr{G}]$.

3. Deligne's problem. Let $\mathscr{F}$ be the free graded Lie algebra over $\boldsymbol{Q}$ generated by the symbols $\tau_m (m : \mathrm{odd} \geq 3)$ of degree m. We have a homomorphism $\psi : \mathscr{F} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \to \mathscr{G} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$ which maps τ_m to σ_m. Deligne proposed the following conjecture:

Conjecture (Deligne). *The homomorphism $\psi : \mathscr{F} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \to \mathscr{G} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$ would be an isomorphism between graded Lie algebras.*

We obtained an affirmative answer for this conjecture in low degrees by a computational method.

Theorem 4. 1. *The conjecture is valid in degrees less than* 13. *Namely, the homomorphism ψ gives an isomorphism*

$$\Big(\mathscr{F} / \bigoplus_{m \geq 13} \mathrm{gr}^m\mathscr{F}\Big) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \simeq \Big(\mathscr{G} / \bigoplus_{m \geq 13} \mathrm{gr}^m\mathscr{G}\Big) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$$

between graded Lie algebras.

2. (*a partial result*) *For $m \leq 15$ and $m = 17$, the homomorphisms*

$$\mathrm{gr}^m\psi : \mathrm{gr}^m\mathscr{F} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \to \mathrm{gr}^m\mathscr{G} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$$

between the m-th degree components are injective.

We shall show the theorem by relating the Galois Lie algebra to a more computable object, the *stable derivation algebra*, and by explicit computation of derivations using computers.

Remark 5. In [4 and 5], Ihara studied the structure of these graded Lie algebras, and

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