On the Asymptotic Behavior of the Occupation Time in Cones of d-dimensional Brownian Motion

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1. Introduction. The purpose of this article is to prove some new feature on the occupation time of Brownian motion. Let $\{B_u\}$ be a d-dimensional Brownian motion starting from the origin and C be a cone with the vertex at the origin. We denote the occupation time of B_u in C by $A_t = \int_0^t 1_C(B_u)du$.

In the only case of $d = 1$ and $C = [0, \infty)$, the explicit law of A_1 is known as Lévy's first arcsin law:

$$P(A_1 \leq s) = \frac{2}{\pi} \arcsin \sqrt{s} \text{ for } 0 \leq s \leq 1.$$

In higher demensions, however, any explicit law of A_1 is not known. We shall present a partial result giving certain asymptotic explicitly. Note that in the one-dimensional case,

$$s^{-\frac{1}{2}} P(A_1 \leq s) \to \frac{2}{\pi} \text{ as } s \to 0.$$

The main result of this article is the following generalization:

Theorem. *Let each of $\{C_i\}_{1 \leq i \leq N}$ be a closed cone with the vertex at the origin and simply connected on S^{d-1} and have $\boldsymbol{C}^2$-class boundary except the origin. Suppose that the cone C is expressed as $C = \cup_{1 \leq i \leq N} C_i$ and satisfies that $C^c \cap S^{d-1}$ is connected on S^{d-1}. Then there exists a positive constant k such that*

$$s^{-\zeta} P(A_1 \leq s) \to k \text{ as } s \to 0,$$

where ζ is defined by $2\zeta = \sqrt{\left(\frac{d}{2} - 1\right)^2 + 2\lambda_1} - \left(\frac{d}{2} - 1\right)$ and λ_1 denotes the first eigenvalue of $-\Delta/2$ on $C^c \cap S^{d-1}$ $(S^{d-1} = \{x \in \boldsymbol{R}^d ; |x| = 1\})$ with Dirichlet boundary condition.

Remark. We should note that in order to make $C^c \cap S^{d-1}$ connected, we need to set $N = 1$ when $d = 2$. In particular, $\zeta = \pi/(2(2\pi - \theta))$ when $d = 2$, where θ denotes the angle of the cone around the origin.

We would like to remark also the following corollary immediately obtained by considering the relation of $\int_0^1 1_{C^c}(B_u)du = 1 - \int_0^1 1_C(B_u)du$.

Corollary. *Suppose that the cone C^c satisfies the same conditions as in the above theorem. Then there exists a positive constant k' such that*

$$(1 - s)^{-\zeta'} P(A_1 \geq s) \to k' \text{ as } s \to 1,$$

where ζ' is defined by λ_1' similarly as in the theorem, and λ_1' denotes the first eigenvalue of $-\Delta/2$ on $C \cap S^{d-1}$ with Dirichlet boundary condition.

There have been some efforts to get its asymptotic behavior. T. Meyre and W. Werner [5] proved that if the cone C is convex, there exist two constants k_1, k_2 depending only on C such that

$$k_1 s^{\zeta} \leq P(A_1 \leq s) \leq k_2 s^{\zeta} \tag{1.1}$$

holds for all $0 \leq s \leq 1$. Recently R. Bass and K. Burdzy [1] proved that if C is a closed cone which satisfies that $C^c \cap S^{d-1}$ is connected and $\nu(\partial C) = 0$ (ν denotes the Lebesgue measure on $\boldsymbol{R}^d$), then we have

$$\lim_{s \to 0} \frac{\log P(A_1 \leq s)}{\log s} = \zeta. \tag{1.2}$$

Both of the above results rely heavily on the estimate of the hitting time to the cone. Since the behavior of the hitting time plays an important role in our proof, we devote Section 2 to investigating it, and in Section 3 we shall give the proof of the main result.

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2. Estimate of the hitting time. In oder to prove our theorem we need a more precise estimate of the hitting time than that employed in the proofs of (1.1) and (1.2). The estimate in Proposition 1 is essential in our proof.

We set $\sigma = \inf\{t \geq 0 ; B_t \in \partial C\}$. Let $\phi(x, y)$ be the angle not exceeding π between ℓ_x and ℓ_y where $\ell_x(\ell_y)$ denotes the line connecting $x(y)$ and the origin. In the case where A is a subset in