

An Extension of Sturm's Theorem to Two Dimensions

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1. Introduction and notation. Let $f(x, y) \in \mathbf{R}[x, y]$ be a square free polynomial with real coefficients, namely $f(x, y)$ is decomposed into the irreducible factors whose multiplicities are only one. Let C be the set of points $(x, y) \in \mathbf{R}^2$ such that $f(x, y) = 0$. Until now, only the following primitive method has been used to draw the curve C by computer, within a given rectangle R . We decompose R into many small rectangles D and obtain $C \cap R$ by gathering $C \cap D$. $C \cap D$ is found as follows.

Let D be the set $\{(x, y) \in \mathbf{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$, and put $P_1 = (a, c)$, $P_2 = (b, c)$, $P_3 = (b, d)$ and $P_4 = (a, d)$. For example, if $f(P_1)f(P_2) < 0$, $f(P_3)f(P_4) < 0$ then we can find approximately a point P_5 in $C \cap \overline{P_1P_2}$ and a point P_6 in $C \cap \overline{P_3P_4}$. Then the line $\overline{P_5P_6}$ can be considered approximately as $C \cap D$.

But the above method has next two problems.

- (1) Even if $f(P_1)f(P_2) > 0$, it is possible that $C \cap \overline{P_1P_2} \neq \emptyset$.
- (2) Even if $C \cap (\text{the boundary of } D) = \emptyset$, it is possible that $C \cap (\text{the interior of } D) \neq \emptyset$.

In this paper, we would like to propose a more reliable method which permit us to liberate from these incertainties.

Let ∂D be the boundary of D and D^i be the interior of D . Then $C \cap D$ is the direct union of $C \cap \partial D$ and $C \cap D^i$. The search for $C \cap D$ is made separately in two cases: the first case for $C \cap \partial D$ and the second case for $C \cap D^i$.

2. First case. This case can be treated as the equation $f = 0$ is restricted to a boundary line. Then we can use Sturm's theorem.

The Sturm sequence associated with the (one-variable) polynomial $f(x)$ is a sequence of polynomials with $f_0(x), f_1(x), \dots, f_k(x)$ defined by the following equations:

$$f_0(x) = f(x), f_1(x) = f'(x),$$

$$f_i(x) = -\text{remainder}(f_{i-2}(x), f_{i-1}(x))$$

where remainder means the remainder from the

division of the former by the latter.

Let (a_1, \dots, a_s) be a sequence of real numbers and (a'_1, \dots, a'_t) be the subsequence of all non-zero numbers. Then $\text{var}(a_1, \dots, a_s)$, the number of sign variations, is the number of i , $1 \leq i < t$, such that $a'_i a'_{i+1} < 0$.

Theorem (Sturm). Let $f(x)$ be a square free polynomial. When $\text{gcd}(f(x), f'(x)) = f_k(x)$, the number of real roots of $f(x)$ in the interval $a < x \leq b$ is

$$\text{var}(f_0(a), f_1(a), \dots, f_k(a)) - \text{var}(f_0(b), f_1(b), \dots, f_k(b)).$$

Let D be the set $\{(x, y) \in \mathbf{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$. Using Sturm's theorem we can determine whether $f(x, c) = 0$ has a root in the interval $[a, b]$ or not. Thus we can determine whether $C \cap \partial D \neq \emptyset$ or not, and if $C \cap \partial D \neq \emptyset$, find this set approximately in considering from divisions of ∂D .

3. Second case. When $C \cap \partial D = \emptyset$ then we can find $C \cap D^i$ in the following manner.

If $C \cap D^i \neq \emptyset$, then there is a point (x_0, y_0) such that $(x_0, y_0) \in C \cap D^i$, but if $(x, y) \in C \cap D^i$, then $y \leq y_0$. Such a point (x_0, y_0) will be called a *maximal point* (of $C \cap D^i$ with respect to y). We write $f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ and show

$f_x(x_0, y_0) = 0$ for a maximal point (x_0, y_0) . If $f_x(x_0, y_0) \neq 0$ then using implicit function theorem, there exists a function $g(y)$ near y_0 such that $f(g(y), y) = 0$ and (x_0, y_0) cannot be a maximal point. Therefore we have $f_x(x_0, y_0) = 0$.

As $f(x, y)$ is square free, we have $\text{gcd}(f(x, y), f_x(x, y)) = 1$ in $\mathbf{R}(y)[x]$. Using Euclidean algorithm we can find $g(x, y), h(x, y) \in \mathbf{R}[x, y]$, $F(y) \in \mathbf{R}[y]$ such that

$$(3) f(x, y)g(x, y) + f_x(x, y)h(x, y) = F(y)$$

If $f(x, y) = 0$, $f_x(x, y) = 0$, then $F(y)$ must be zero. Using Sturm's theorem, we can count correctly the number of roots $F(y) = 0$ in the interval $[c, d]$ and we can calculate approximately all roots in this interval. Therefore we can calculate