On the Number of Asymptotic Points of Holomorphic Curves

By Nobushige TODA

Department of Mathematics, Nagoya Institute of Technology

(Communicated by Kiyosi ITÔ, M. J. A., Dec. 12, 1997)

1. Introduction. Let $f = [f_1, \ldots, f_{n+1}]$ be a transcendental holomorphic curve from $\boldsymbol{C}$ into the n dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1, \ldots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\mathbf{O}\},$$

where n is a positive integer.

We use the following notation:

$$\|f(z)\| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{1/2}$$

and for a point $\boldsymbol{a} = (a_1, \ldots, a_{n+1})$ in $\boldsymbol{C}^{n+1} - \{\mathbf{O}\}$

$$\|\boldsymbol{a}\| = (|a_1|^2 + \cdots + |a_{n+1}|^2)^{1/2},$$

$$(\boldsymbol{a}, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1},$$

$$(\boldsymbol{a}, f(z)) = a_1 f_1(z) + \cdots + a_{n+1} f_{n+1}(z),$$

$$d(\boldsymbol{a}, f(z)) = |(\boldsymbol{a}, f(z))| / (\|\boldsymbol{a}\| \, \|f(z)\|).$$

(On the distance "d", see [7], p. 76, where $\| \; \|$ is used instead of d).

The characteristic function $T(r, f)$ of f is defined as follows (see [7]):

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta - \log \|f(0)\|.$$

We note that

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = \infty$$

since f is transcendental.

We put

$$\rho = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

$$\lambda = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and we say that ρ is the order of f and λ the lower order of f.

Let

$$V = \{\boldsymbol{a} \in \boldsymbol{C}^{n+1} : (\boldsymbol{a}, f) = 0\}.$$

Then, V is a subspace of $\boldsymbol{C}^{n+1}$ and $0 \leq \dim V \leq n - 1$. It is said that f is linearly nondegenerate when $\dim V = 0$ and linearly degenerate otherwise.

For meromorphic functions in $|z| < \infty$ we shall use the standard notation and symbols of the Nevanlinna theory of meromorphic functions ([2]).

For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - V$, we put

$$N(r, \boldsymbol{a}, f) = N(r, 1/(\boldsymbol{a}, f))$$

and we denote the standard basis of $\boldsymbol{C}^{n+1}$ by $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n+1}$.

Let X be a subset of $\boldsymbol{C}^{n+1}$. Then, we say that X is **in general position** if the elements of X are linearly independent when $\# X \leq n$ or if any $n + 1$ elements of X are linearly independent when $\# X \geq n + 1$.

The purpose of this paper is to extend a famous result on the number of asymptotic values of meromorphic functions obtained by Ahlfors in [1] to holomorphic curves. By the way, the result in [1] was extended to algebroid functions by Lü Yinian in [5].

2. Definition and lemma. In this section, we first give a definition of asymptotic point to holomorphic curves. Let f be as in Section 1.

Definition 1 (asymptotic point) (see Definition 3 in [6]). A point $\boldsymbol{a}$ of $\boldsymbol{C}^{n+1} - V$ is an asymptotic point of f if and only if there exists a path Γ: $z = z(t)$ $(0 \leq t < 1)$ in $|z| < \infty$ satisfying the following conditions:

(i) $\lim_{t\to 1} z(t) = \infty$;

(ii) $\lim_{t\to 1} d(\boldsymbol{a}, f(z(t))) = 0$.

Remark. This definition is a generalization of "asymptotic values" of meromorphic functions.

In fact, let $g = g_2/g_1$ be a transcendental meromorphic function in $|z| < \infty$, where g_1 and g_2 are entire functions without common zeros. Suppose that g has an asymptotic value c along a path L going from a finite point to ∞ and put $\tilde{g} = [g_1, g_2]$.

(i) When $c \neq \infty$, for $\boldsymbol{a} = (-c, 1) \in \boldsymbol{C}^2$,

$$d(\boldsymbol{a}, \tilde{g}(z)) = \frac{|-cg_1(z) + g_2(z)|}{\|\boldsymbol{a}\|(|g_1(z)|^2 + |g_2(z)|^2)^{1/2}} = \frac{|g(z) - c|}{\|\boldsymbol{a}\|(1 + |g(z)|^2)^{1/2}} \to 0$$

as $z \to \infty$ along L;

(ii) when $c = \infty$, for $\boldsymbol{e}_1 \in \boldsymbol{C}^2$,

$$d(\boldsymbol{e}_1, \tilde{g}(z)) = \frac{|g_1(z)|}{(|g_1(z)|^2 + |g_2(z)|^2)^{1/2}} = \frac{1}{(1 + |g(z)|^2)^{1/2}} \to 0$$