A Note on the Diophantine Equation $a^x + b^y = c^z$

By Nobuhiro TERAI*) and Kei TAKAKUWA**)

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§ 1. **Introduction.** In our previous papers Terai [6], [7] and [8], we proposed the following conjecture and proved it under some conditions when $p = 2$, $q = 2$ and r is an odd prime.

Conjecture. *If a, b, c, p, q, r are fixed positive integers satisfying $a^p + b^q = c^r$ with $p, q, r \geq 2$ and $(a, b) = 1$, then the Diophantine equation*

$$(1) \qquad a^x + b^y = c^z$$

has only the positive integral solution $(x, y, z) = (p, q, r)$.

The positive integers a, b, c satisfying $a^2 + b^2 = c^r$ can be expressed as follows (cf. Lemma 1 in [8]):

Lemma 1. *The positive integral solutions of the equation $a^2 + b^2 = c^r$ with $(a, b) = 1$ and r odd ≥ 3 are given by*

$$a = \pm u \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j} u^{r-(2j+1)} v^{2j},$$

$$b = \pm v \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j+1} u^{r-(2j+1)} v^{2j},$$

$c = u^2 + v^2$, *where u, v are integers such that $(u, v) = 1$ and $u \not\equiv v \pmod 2$.*

From now on, let a, b, c be as in Lemma 1 with $u = m$, $v = 1$; i.e.

$$(2) \quad a = \pm m \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j} m^{r-(2j+1)},$$

$$b = \pm \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j+1} m^{r-(2j+1)}, \; c = m^2 + 1,$$

where m is a positive integer with $2|m$.

Then in [6], [7] and [8], we showed that if b is an odd prime and there is an odd prime l such that $ab \equiv 0 \pmod l$ and $e \equiv 0 \pmod r$, where e is the order of c modulo l, then equation (1) has only the positive integral solution $(x, y, z) = (2, 2, r)$ under some conditions. Recently, using the divisibility property concerning Lucas seqences, when $r = 3$, Le [3] has proved the following:

Theorem. (Le [3]). *Let a, b, c be positive integers satisfying (2) with $r = 3$. If $2 \| m$ and b is an odd prime, then equation (1) has only the positive integral solution $(x, y, z) = (2, 2, 3)$.*

In this paper, using a similar method as in [3], when r is an odd prime, we generalize Le's theorem as follows:

Theorem 1. *Let r be an odd prime. Let a, b, c be positive integers satisfying (2). Let m be a positive integer with $2 \| m$ and $m \geq 6$. Suppose that b is an odd prime and b satisfies at least one of the following three conditions:*

(i) $b \equiv -1 \pmod m$, (ii) $b \equiv -1 \pmod 4$,
(iii) $\left(\frac{b}{a'}\right) = -1$,

where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol and $a = ma'$.

Then equation (1) has only the positive integral solution $(x, y, z) = (2, 2, r)$.

In Theorem 1, we suppose that $2 \| m$ and $m \geq 6$. When $m = 2$, we also prove the following theorem. We note that we need not suppose b is an odd prime in Theorem 2.

Theorem 2. *Let r be an odd prime. Let a, b, c be positive integers satisfying (2) with $m = 2$. Suppose that b satisfies at least one of the following three conditions:*

(i) $b \equiv -1 \pmod 3$, (ii) $b \equiv -1 \pmod 4$,
(iii) $\left(\frac{b}{a'}\right) = -1$,

where $a = 2a'$.

Then equation (1) has only the positive integral solution $(x, y, z) = (2, 2, r)$.

Since $b = 3m^2 - 1 \equiv -1 \pmod 4$ when $r = 3$, Theorems 1 and 2 give a generalization of Le's theorem.

§ 2. **Lemmas.** In this section, we prepare some lemmas used in the proof of Theorems 1 and 2.

Lemma 2. *Let r be odd ≥ 3. Let a, b, c be positive integers satisfying (2). Let m be a positive integer with $2 \| m$ and $m \geq 6$. Suppose that b satis-*

*) Division of General Education, Ashikaga Institute of Technology.

**) Department of Mathematics, Faculty of Science, Gakushuin University.