On the Polynomial Hamiltonian Structure Associated with the $L(1, g+2; g)$ Type

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1. Introduction. In the paper [3], we considered the differential equation

$$(1.1)\quad \frac{d}{dx}Y = \frac{1}{x}\mathscr{A}(x, t)Y,\ \mathscr{A}(x, t) := \sum_{k=0}^{g+1}\mathscr{A}_k x^k,$$

which satisfies the conditions:

(i) $\mathscr{A}_k$ are 2×2 matrices,

(ii) the eigenvalues of $\mathscr{A}_0$ are distinct up to additive integers,

(iii) the eigenvalues of $\mathscr{A}_{g+1}$ are distinct.

This equation can be reduced to the equation

$$(1.2)\quad \frac{d^2}{dx^2}y + p_1(x, t)\frac{d}{dx}y + p_2(x, t)y = 0,$$

which satisfies the three conditions:

(iv) The Riemann scheme of (1,1) is

$$\left\{\begin{matrix} x=0 & x=\lambda_1 & \cdots & x=\lambda_g & & & & & & x=\infty & & \\ 0 & 0 & \cdots & 0 & & 0 & 0 & 0 & \cdots & 0 & & -\kappa_\infty \\ \kappa_0 & 2 & & 2 & & \frac{1}{g+1} & \frac{t_g}{g} & \frac{t_{g-1}}{g-1} & \cdots & t_1 & & \kappa_\infty - \kappa_0 + 1 \end{matrix}\right\},$$

(v) κ_0 and κ_∞ are not integer,

(vi) $x = \lambda_k$ $(k = 1, \ldots, g)$ are non-logarithmic singular points.

The equation (1.2) is called $L(1, g+2; g)$ type.

Let μ_k $(k = 1, \ldots, g)$ be the residue of $p_2(x, t)$ at $x = \lambda_k$ and let

$$h_j := \frac{\partial^{g-j}}{\partial x^{g-j}}\left(xp_2 - \sum_{k=1}^{g}\frac{\lambda_k\mu_k}{x-\lambda_k}\right)\Bigg|_{x=0}.$$

By the assumtion (vi), we remark that these h_j are uniquely determined as rational functions in $\lambda_k, \mu_k, t_k (k = 1, \ldots, g)$.

Using these notations λ_k, μ_k and h_k, we state that the holonomic deformation of the linear equation (1.2) is governed by the Hamiltonian system:

$$\frac{\partial\lambda_i}{\partial t_j} = \frac{\partial\tilde{K}_j}{\partial\mu_i},\ \frac{\partial\mu_i}{\partial t_j} = -\frac{\partial\tilde{K}_j}{\partial\lambda_i}\ (i, j = 1, \ldots, g),$$

where the Hamiltonian $\tilde{K}_j$ are

$$\begin{bmatrix}\tilde{K}_1\\ \tilde{K}_2\\ \vdots\\ \vdots\\ \tilde{K}_g\end{bmatrix} = \begin{bmatrix}1 & & & & 0\\ & 2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & g\end{bmatrix}^{-1} \begin{bmatrix}1 & & & & 0\\ t_g & 1 & & & \\ \vdots & t_g & 1 & & \\ t_3 & & \ddots & \ddots & \\ t_2 & t_3 & \cdots & t_g & 1\end{bmatrix}^{-1} \begin{bmatrix}h_1\\ h_2\\ \vdots\\ \vdots\\ h_g\end{bmatrix}.$$

It is known that, if $g = 1$ the holonomic deformation of (1.2) is governed by the fourth Painlevé equation. So the Hamiltonian system $(\lambda, \mu, \tilde{K}, t)$ is a extension of the fourth Painlevé equation.

Now, we assume that $g < 8$. The purpose of this paper is to transform the Hamiltonian system $(\lambda, \mu, \tilde{K}, t)$ into the Hamiltonian system (q, p, H, ξ) with the conditions:

(C_1) H_j $(j = 1, \ldots, g)$ are polynomial in q_k, p_k $(k = 1, \ldots, g)$,

(C_2) $\dfrac{\partial H_k}{\partial \xi_j} = \dfrac{\partial H_j}{\partial \xi_k}$.

We will state that if $\kappa_\infty = 0$, a special solution of this system is written by the multivalue Hermite function.

By the condition (C_2), we can introduce the function $\tau_{\mathrm{IV}}^{(g)}$ as follows:

$$\frac{\partial}{\partial\xi_k}\log\tau_{\mathrm{IV}}^{(g)} = H_k.$$

In [7], Okamoto defined the τ function associate with the fourth Painlevé transcendental function. This function is equivalent to $\tau_{\mathrm{IV}}^{(1)}$.

2. Polynomial Hamiltonian structure.

Theorem 2.1. *Put σ_k, ρ_k $(k = 1, \ldots, g)$ as follows:*

σ_k = the k-th elementary symmetric function of $\lambda_1, \lambda_2, \ldots, \lambda_g$,

$$\rho_k = (-1)^{k-1}\sum_{l=1}^{g}\lambda_l^{g-k}\mu_l\prod_{\substack{j=1\\ j\neq 1}}(\lambda_l - \lambda_j)^{-1}.$$