On the Holonomic Deformation of Linear Differential Equations

By Hiroyuki KAWAMUKO

Department of Mathematical Sciences, University of Tokyo

(Communicated by Kiyosi ITÔ, M. J. A., Oct. 13, 1977)

1. Introduction. Consider a linear differential equation:

$$\frac{d}{dx}Y = M(x, t)Y, \tag{1.1}$$

where $M(x, t)$ is an $m \times m$ matrix whose entries are rational functions of x, and depend on $t \in U \subset \mathbf{C}^g$ analytically. We call the following problem as extended Fuchs problem. "Give a condition under which there exist a solution whose monodromy groups and Stokes multipliers are independent of t".

When the differential equation (1.1) is of the form:

$$\frac{d}{dx}Y = \mathscr{A}(x, t)Y,$$

$$\mathscr{A}(x, t) = \sum_{j=1}^{n}\sum_{k=0}^{r_j}\frac{\mathscr{A}_{j,-k}}{(x-a_j)^{k+1}} - \sum_{k=1}^{r_\infty}\mathscr{A}_{\infty,-k}x^{k-1},$$

the Fuchs problem was studied by Jimbo-Miwa-Ueno [4 and 6]. They show that a solution of this problem is given by a nonlinear differential equation with the Painlevé property. This nonlinear differential equation is called the monodromy preserving deformation equation, called in short MPD equation.

It is known by [7, 8, and 9] that the Garnier system and the Painlevé equations are special cases of MPD equation, and that each of these equations is described as a polynomial Hamiltonian systems. By the use of these results, the contiguity relations of Painlevé equations are given by [10, 11, 12, and 13].

In this paper, we consider the Fuchs problem for the linear differential equation:

$$\frac{d}{dx}Y = \frac{1}{x}\mathscr{A}(x, t)Y, \quad \mathscr{A}(x, t) := \sum_{k=0}^{g+1}\mathscr{A}_k x^k, \tag{1.2}$$

with following assumptions

(i) $\mathscr{A}_k$ are 2×2 matrices,

(ii) the eigenvalues of $\mathscr{A}_0$ are distinct up to additive integers,

(iii) the eigenvalues of $\mathscr{A}_{g+1}$ are distinct.

We show in what follows that the MPD equation is written as a Hamiltonian system. Notice that if $g = 1$, the MPD equation is equivalent to the fourth Painlevé equation, and that if $g = 2$, the MPD equation is equivalent to the nonlinear differential equation given in [5].

2. Holonomic deformation. Theorem 2.1. *Changing suitably the variables, we can transform* (1.2) *to the linear differential equation*:

$$\frac{d}{dx}Y = \frac{1}{x}\sum_{k=0}^{g+1}\bar{\mathscr{A}}_k Y, \tag{2.1}$$

which satisfies the following conditions:

- $\bar{\mathscr{A}}_{g+1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, • $\det \bar{\mathscr{A}}_0 = 0$,
- $\deg \sum_{k=0}^{g+1} \bar{\mathscr{A}}_k x^k \le g+1$, • *The* (1,1) *component of* $\bar{\mathscr{A}}_g$ *is* 0.

Theorem 2.2. *The differential equation* (2.1) *is equivalent to the following equation*:

$$\frac{d^2}{dx^2}y + p_1(x, t)\frac{d}{dx}y + p_2(x, t)y = 0 \tag{2.2}$$

$$p_1(x, t) = \frac{1-\kappa_0}{x} - \sum_{k=1}^{g} t_k x^{k-1} - x^g - \sum_{k=1}^{g}\frac{1}{x-\lambda_k},$$

$$p_2(x, t) = -\frac{1}{x}\sum_{k=1}^{g} h_{g+1-k}\cdot x^{k-1} + \kappa_\infty x^{g-1} + \sum_{k=1}^{g}\frac{\lambda_k\mu_k}{x(x-\lambda_k)},$$

where h_k $(k = 1, \ldots, g)$ *are*

$$h_j = (-1)^{j-1}\sum_{l=1}^{g}\frac{1}{\Lambda'(\lambda_l)}\Big[\lambda_l\sigma_{l,j-1}\mu_l^2 - \sigma_{l,j-1}(\lambda_l^{g+1} + \sum_{k=1}^{g} t_k\lambda_l^k + \kappa_0)\mu_l + \kappa_\infty\lambda_l^g\sigma_{l,j-1}\Big] - \sum_{l=1}^{g}\sum_{k=0}^{j-2}(-1)^k\sigma_{l,k}\lambda_l^{j-1-k}\frac{\mu_l}{\Lambda'(\lambda_l)},$$

$$\Lambda(\lambda_k) = \prod_{\substack{j=1\\ j\neq k}}^{g}(\lambda_k - \lambda_j),$$

$$\sigma_{k,j} = \frac{1}{j!}\frac{d^j}{dx^j}\prod_{\substack{i=1\\ i\neq k}}^{g}(1+\lambda_i x)\Big|_{x=0}.$$

The number of accessory parameters (2.2) is $2g$. (2.2) has singular points at $x = 0$ and $x =$