Transcendence of Rogers-Ramanujan Continued Fraction and Reciprocal Sums of Fibonacci Numbers

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The Rogers-Ramanujan continued fraction $RR(q)$ is defined by

$$RR(q) = 1 + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} + \cdots,$$

which is known to have the expansions

$$RR(q) = \frac{\sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)}}{\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(1-q)(1-q^2)\dots(1-q^k)}} = \prod_{k=0}^{\infty} \frac{(1-q^{5k+2})(1-q^{5k+3})}{(1-q^{5k+1})(1-q^{5k+4})}$$

(cf. [2 ; (3.4.9)]). Irrationality measures were given by Osgood [8] and Shiokawa [9]. It is proved in [9] that, for any integer $d \geq 2$, there is a constant $C = C(d) > 0$ such that

$$\left| RR\left(\frac{1}{d}\right) - \frac{p}{q} \right| > Cq^{-2-B/\sqrt{\log q}}$$

for all integers p, q (≥ 2), where $B = \sqrt{\log d}$. Matala-Aho [5] obtained some higher degree irrationality results. An example of Theorem 1 in [5] is $RR((\sqrt{5}-1)/2) \notin \boldsymbol{Q}(\sqrt{5})$.

In this note we first prove the following.

Theorem 1. The Rogers–Ramanujan continued fraction $RR(q)$ is transcendental for any algebraic number q with $0 < |q| < 1$.

The proof is a simple application of Lemma 1 and 2 below, which are proved in the same manner as in [3]. Lemma 2 is a straightforward consequence of a recent theorem of Nesterenko on modular functions ([6] and [7]).

As usual we set for $|q| < 1$

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

$$E_4(q) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

$$E_6(q) = 1 - 540\sum_{n=1}^{\infty} \sigma_5(n)q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$,

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

and

$$\theta_3 = 1 + 2\sum_{n=1}^{\infty} q^{n^2},\ \theta = \theta_4 = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2},$$

$$\theta_2 = 2q^{1/4}\sum_{n=1}^{\infty} q^{n(n-1)}.$$

Let $\boldsymbol{K} = \boldsymbol{Q}(E_2, E_4, E_6)$.

Lemma 1 ([4]). Let $y = y(q)$ denote any one of θ_3, θ_4, and θ_2. Then the functions $\eta(q^k)$, $\eta'(q^k)$, $\eta''(q^k)$, $y(q^k)$, $y'(q^k)$, and $y''(q^k)$ are algebraic over $\boldsymbol{K}$ for every positive integer k, where ′ denotes the derivation $q\frac{d}{dq}$.

Lemma 2 ([4]). Suppose that α is an algebraic number with $0 < |\alpha| < 1$. If a nonconstant function f is algebraic over $\boldsymbol{K}$ and defined at α, then $f(\alpha)$ is transcendental.

Proof of Theorem 1. Let

$$F(q) = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} + \cdots,$$

then

$$\frac{1}{F(q)} - F(q) - 1 = q^{1/5}\frac{\prod_{n=1}^{\infty}(1-q^{n/5})}{\prod_{n=1}^{\infty}(1-q^{5n})} = q^{2/5}\frac{\eta(q^{1/5})}{\eta(q^5)}$$

(see [1 ; p. 85]). Applying Lemma 1 and 2 to the function $f(q) = \eta(q)/\eta(q^{25})$, we see that, for any algebraic number q with $0 < |q| < 1$, $f(q)$ is transcendental, and so is $F(q)$ from the formula above.

Now we give further examples of continued fractions whose transcendence can be easily deduced from Lemma 1 and 2. For any algebraic number q with $0 < |q| < 1$, the following con-

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