Armendariz Rings

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1. Introduction. Let R be a domain (commutative or not) and $R[x]$ its polynomial ring. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ be elements of $R[x]$. (This notation for the coefficients of $f(x)$ and $g(x)$ will be followed in the absence of explicit mention.) It is an elementary exercise to prove that if $f(x)g(x) = 0$, then $a_i b_j = 0$ for every i and j, since either $f(x) = 0$ or $g(x) = 0$. (Of course the converse always holds.)

E. Armendariz ([1], Lemma 1) noted that the above result can be extended to the class of reduced rings, i.e., rings without non-zero nilpotent elements. In order to study additional classes of rings having this property we introduce the following definition.

1.1. Definition. A ring R is said to have the Armendariz property (or is an *Armendariz ring*) if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)\,g(x) = 0$, we have $a_i b_j = 0$ for every i and j.

By a ring we mean an associative ring with identity. However, the assumption of the existence of identity can be omitted in many places. Many remarks are thus valid in the context of "rings" and subrings (i.e., subrings which may not inherit the identity of the over-ring). For defining left/right zero-divisors, we shall refer to ([4], p. 88).

In addition to reduced rings, there are large classes of rings which are Armendariz. If R is a commutative P.I.D and A an ideal of R, then R/A is Armendariz (Theorem 2.2). If K is a field and V is a vector space over K, then the ring K $(+)$ V (see 1.2 for notation) is an Armendariz ring (Corollary 2.9).

For constructing examples of both Armendariz rings and non-Armendariz rings, we shall use the following principle of idealisation due to Nagata ([6], p.2).

1.2. Let R be a commutative ring and M an R-module. The R-module $R \oplus M$ acquires a ring structure where the product is defined by

$$(a, m)(b, n) = (ab, an + bm).$$

We shall use the notation R $(+)$ M for this ring. If M is not zero, this ring is not reduced, since M can be identified with the ideal $0 \oplus M$ which has square zero. (It seems appropriate to call this ring as "R Nagata M").

We shall also need the following variants of the construction in 1.2.

1.3. Let R be a commutative ring and h: $R \to R$ a ring homomorphism. Let M be an R-module. On modifying the definition in 1.2 to

$$(a, m)(b, n) = (ab, h(a)n + bm),$$

we get a (non-commutative) ring structure on $R \oplus M$ which we shall denote by $R\ (+)_h M$.

1.4. Let R be a ring and A an ideal of R. The factor ring $\bar{R} = R/A$ has the natural structure of a left R-, right R- bimodule. Denote $\bar{a} = a + A \in \bar{R}$ for each $a \in R$. We use this structure to define a ring structure on $R \oplus (R/A)$ as follows:

$$(r, \bar{a})(r', \bar{a}') = (rr', \overline{ra' + ar'}).$$

We denote this ring by R $(+)$ R/A. Its properties are similar to those of R $(+)$ M.

2. Rings which have the Armendariz property. It is easy to see that subrings of Armendariz rings are also Armendariz. However, factor rings need not be so (see 3.3). If $\{R_i\}_{i \in I}$ are Armendariz, so is ΠR_i. We begin with examples of familiar non-reduced rings which are Armendariz.

2.1. Proposition. For each integer n, $\boldsymbol{Z}/n\boldsymbol{Z}$ is an Armendariz ring, which is not reduced whenever n is a natural number which is not square free.

Proof. We first consider the case $n = p^m$, p a prime. Denote by $\overline{f(x)}$, $\overline{g(x)}$ the cosets of $f(x)$, $g(x)$ $(\mathrm{mod}\, p^m \boldsymbol{Z}[x])$, respectively. Assume $\overline{f(x)g(x)} = 0$, i.e. $p^m \mid f(x)g(x)$. Since p is a prime, it follows that $f(x) = p^r f'(x)$ and $g(x) = p^s g'(x)$ for some f' and g' satisfying the conditions that the $g.\ c.\ d.$ of the coefficients of f' (also of g') is not