On the Zeros of $\sum a_i expg_i$ *)

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Abstract: We consider entire functions of the form $f = \sum a_i e^{g_i}$, where $a_i (\not\equiv 0)$, g_i are entire functions and the orders of all a_i are less than one. If all the zeros of f are real, then $f = e^g \sum a_i e^{h_i}$, where h_i are linear functions. Using this result, we can prove that $f = a_1 e^g$ if all zeros of f are positive, which also generalizes a result obtained by A. Eremenko and L. A. Rubel.

Key words: Zero set; entire function; Borel theorem; upper half-plane; Nevanlinna theory.

1. Introduction and main results. For $i \geq 1$ and $z \in \mathrm{C}$, let $g_i(z)$ be entire functions. Let $a_i(z)$ be a non-zero entire function with $\rho(a_i) < 1$, where $\rho(g)$ denotes the order of an entire function g. Let B_1 denote the class of entire functions of the form

$$f = \sum_{i=1}^{n} a_i e^{g_i},$$

where $e^{g_i - g_j}$ is non-constant for $i \neq j$.

If all the a_i are polynomials, then such f is said to be in the class B. Clearly, B is a proper subset of B_1.

Let $Z(g)$ be the zero set of an entire function g. In [2], by using H. Cartan's theory of holomorphic curves. A. Eremenko and L. A. Rubel proved the following theorem.

Theorem A. Let $f \in B$. If $Z(f)$ is a subset of the positive real axis, except possibily finitely many points, then $f = pe^g$, where p is a polynomial and g is an entire function.

Therefore, it is natural to ask whether we can say something about the form of f if $f \in B$ and $Z(f)$ is a subset of the real axis. By adapting some of the arguments used in [6] and Nevanlinna value distribution theory for functions meromorphic in a half plane, we can answer this question even for the case $f \in B_1$. In fact, we obtained the following results.

Theorem 1. *Let $f \in B_1$. If $Z(f)$ is a subset of the real axis, except possibly finite points, then $f(z) = e^{g(z)} \sum_{i=1}^{n} a_i(z) e^{b_i z}$, where $b_i \in \mathrm{C}$, g and $a_i (\not\equiv 0)$ are entire functions with $\rho(a_i) < 1$.*

Using theorem 1, we can generalize theorem A to the following theorem.

Theorem 2. *Let $f \in B_1$. If $Z(f)$ is a subset of the positive real axis, except possibly finite points, then $f = ae^g$, where g, a are entire functions with $\rho(a) < 1$.*

Our basic tool is J. Rossi's half-plane version of Borel theorem. J. Rossi proved this version in [6] by using Tsuji's half-plane version of Nevanlinna theory. Therefore, we shall start with the basic notations of Tsuji's theory (cf. [4] and [7]); assuming the readers are familiar with the Nevanlinna Theory and its basic notations (cf. [3]).

Let $n_u(t, \infty)$ be the number of poles of f in $\{z : |z - \frac{it}{2}| \leq \frac{t}{2}, |z| \geq 1\}$, where f is meromorphic in the open upper half-plane. Define

$$N_u(r, \infty) = N_u(r, f) = \int_1^r \frac{n_u(t, \infty)}{t^2} dt,$$

$$m_u(r, \infty) = m_u(r, f)$$
$$= \frac{1}{2\pi} \int_{arcsinr^{-1}}^{\pi - arcsinr^{-1}} log^+ |f(rsin\theta e^{i\theta})| \frac{d\theta}{rsin^2\theta},$$

$$N_u(r, a) = N_u(r, \frac{1}{f-a}), \; m_u(r, a)$$
$$= m_u(r, \frac{1}{f-a}) \; (a \neq \infty) \text{ and}$$
$$T_u(r, f) = m_u(r, f) + N_u(r, f).$$

Remark 1. We can also define $m_l(r, f)$, $N_l(r, f)$, $T_l(r, f)$ for functions meromorphic in the open lower half-plane in the obvious way.

Lemma 1 [4]. *Let f be meromorphic in Imz*

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