Stationary Solutions of the Heat Convection Equations in Exterior Domains

By Kazuo ŌEDA

Faculty of Science, Japan Women's University

(Communicated by Kiyosi ITÔ, M. J. A., June 12, 1997)

1. Introduction. Let $\Omega = K^c \subset \boldsymbol{R}^3$ where K is a compact set whose boundary ∂K is of class C^2. We put $\partial\Omega = \Gamma = \partial K$. Then we consider the stationary problem for the heat convection equation (HCE) in Ω :

$$(1) \quad \begin{cases} (u \cdot \nabla)u = -(\nabla p)/\rho \\ \qquad + \{1 - \alpha(\theta - \Theta_0)\}g + \nu\Delta u & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ (u \cdot \nabla)\theta = \kappa\Delta\theta & \text{in } \Omega, \end{cases}$$

$$(2) \quad u|_\Gamma = 0, \quad \theta|_\Gamma = \Theta_0 > 0, \quad \lim_{|x|\to\infty} u(x) = 0, \quad \lim_{|x|\to\infty} \theta(x) = 0,$$

where $u = u(x)$ is the velocity vector, $p = p(x)$ is the pressure and $\theta = \theta(x)$ is the temperature; ν, κ, α, ρ, and $g = g(x)$ are the kinematic viscosity, the thermal conductivity, the coefficient of volume expansion, the density at $\theta = \Theta_0$ and the gravitational vector, respectively.

As concerns the exterior problem of (HCE), Hishida [2] proved the global existence of the strong solution of the initial value problem (IVP) in the case that K is a ball. Recently, Ōeda-Matsuda [10] showed the existence and uniqueness of weak solutions of (IVP) when K is a compact set with the boundary of class C^2. In [10], the approach to prove the existence of weak solutions was "the extending domain method", that is, the exterior domain Ω was approximated by interior domains $\Omega_n = B_n \cap \Omega$ (B_n is a ball with radius n and center at O) as $n \to \infty$ (see Ladyzhenskaya [3]). On the other hand, Morimoto [6], [7] studied the stationary problem of (HCE) in interior domains and showed the existence and uniqueness of weak solutions. The purpose of the present paper is to show the existence of stationary weak solutions of (HCE) by using "the extending domain method". Moreover, we also study the uniqueness of a weak solution.

2. Preliminaries. We make several assumptions (A1)~(A3):

(A1) $\omega_0 \subset \operatorname{int} K$ (ω_0 is a neighbourhood of the origine O) and $K \subset B = B(O, d)$ which is a ball with radius d and center at O. (A2) $\partial\Omega = \Gamma = \partial K \in C^2$. (A3) $g(x)$ is a bounded and continuous vector function in $\boldsymbol{R}^3 \setminus \omega_0$. Moreover there exist $R_0 > 0$, $C_{R_0} > 0$ such that $|g| \leq C_{R_0}/|x|^{\frac{5}{2}+\varepsilon}$ for $|x| \geq R_0$ ($\varepsilon > 0$ is arbitrary).

Remark 1. By (A3), we can take $C_w > 0$ such that $|g| \cdot |x|^{\frac{5}{2}+\varepsilon} \leq C_w$ for $x \in \boldsymbol{R}^3 \setminus \omega_0$. Moreover $g \in L^p(\Omega)$ for $p \geq \dfrac{6}{5}$.

Here, in order to transform the boundary condition on θ to a homogenuous one, we introduce an auxiliary function $\bar{\theta}$ (see [1] p.131, [11] p.175):

Lemma 2.1. *There exists a function* $\bar{\theta}$ *which satisfies the following properties* (i) ~ (iii): (i) $\bar{\theta}(\Gamma) = \Theta_0$. (ii) $\bar{\theta} \in C_0^2(\boldsymbol{R}^3)$. (iii) *For any* $\varepsilon > 0$ *and* $p \geq 1$, *we can retake* $\bar{\theta}$, *if necessary, such that* $\|\bar{\theta}\|_{L^p} < \varepsilon$.

Now we make a change of variable: $\theta = \hat{\theta} + \bar{\theta}$. And after changing of variable, we use the same letter θ. The system of equations (1) and (2) is transformed to the following:

$$(3) \quad \begin{cases} (u \cdot \nabla)u = -(\nabla p)/\rho - \alpha\theta g + \nu\Delta u \\ \qquad + \{1 - \alpha(\bar{\theta} - \Theta_0)\}g & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ (u \cdot \nabla)\theta = \kappa\Delta\theta - (u \cdot \nabla)\bar{\theta} + \kappa\Delta\bar{\theta} & \text{in } \Omega, \end{cases}$$

$$(4) \quad u|_\Gamma = 0, \quad \theta|_\Gamma = 0, \quad \lim_{|x|\to\infty} u(x) = 0, \quad \lim_{|x|\to\infty} \theta(x) = 0.$$

We use several function spaces. G denotes Ω or Ω_n.

$W^{k,p}(G) = \{u\,;\, D^\alpha u \in L^p(G),\ |\alpha| \leq k\}$, $W_0^{k,p}(G) =$ the completion of $C_0^k(G)$ in $W^{k,p}(G)$, $D_\sigma(G) = \{\varphi \in C_0^\infty(G)\,;\, \operatorname{div}\varphi = 0\}$, $D(G) = \{\psi \in C_0^\infty(G \cup \Gamma)\,;\, \psi(\Gamma) = 0\}$, $H_\sigma(G)$ (resp. $H_\sigma^1(G)$) = the completion of $D_\sigma(G)$ in $L^2(G)$ (resp. $W^1(G)$), V (resp. W) = the completion of $D_\sigma(\Omega)$ (resp. $D(\Omega)$) in $\|\cdot\|_{N(\Omega)}$, where $\|u\|_{N(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$, $H_0^1(\Omega_n)$ = the completion of $D(\Omega_n)$ in $W^{1,2}(\Omega_n)$ (it turns out $H_0^1(\Omega_n) = W_0^{1,2}(\Omega_n)$).