On the Vanishing of Iwasawa Invariants of Certain Cyclic Extensions of $\boldsymbol{Q}$ with Prime Degree

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(Communicated by Shokichi IYANAGA, M. J. A., June 12, 1997)

Abstract: Let ℓ be an odd prime. In [2], Yamamoto gave a condition for (ℓ, ℓ)-extensions K of $\boldsymbol{Q}$, under which the Iwasawa invariants $\lambda_\ell(K)$ and $\mu_\ell(K)$ vanish. In this note, we shall give a condition for (ℓ, ℓ)-extensions K of $\boldsymbol{Q}$, which is weaker than the condition given in [2], under which we have $\lambda_\ell(k) = \mu_\ell(k) = 0$ for any subfields k of K with $[k:\boldsymbol{Q}] = \ell$. Our proof is based on Greenberg's original idea (cf. [1]), which is more elementary than that in [2], using the capitulation of the ℓ-part of the ideal class group of k in the initial layer of the cyclotomic $\boldsymbol{Z}_\ell$-extension of k to assure $\lambda_\ell(k) = \mu_\ell(k) = 0$.

Key words: Capitulation; Iwasawa invariants.

1. Introduction. Throughout the paper, we fix an odd prime number ℓ. For a cyclic extension k of $\boldsymbol{Q}$ of degree ℓ, we denote by $A(k)$ the ℓ-primary part of the ideal class proup of k and $B(k)$ the subgroup of $A(k)$ consisting of elements which are invariant under the action of the Galois group $G(k/\boldsymbol{Q})$. Let $p_1, p_2, \cdots, p_s$ be the prime numbers which are ramified in $k/\boldsymbol{Q}$ and let $\mathfrak{p}_i$ be the prime ideal of k lying over p_i. Then it is easy to see from the genus theory that $B(k)$ is an ℓ-elementary abelian group of rank $s-1$ generated by $\mathrm{cl}(\mathfrak{p}_1), \mathrm{cl}(\mathfrak{p}_2), \cdots, \mathrm{cl}(\mathfrak{p}_s)$. Let $\bar{k}$ (resp. $\tilde{k}$) be the ℓ-part of the Hilbert class field (resp. genus field) of k. Then we have the isomorphism $A(k) \xrightarrow{\sim} G(\bar{k}/k)$ and hence the surjective homomorphism

$$\varphi : A(k) \ni \mathrm{cl}(\mathfrak{a}) \mapsto \left(\frac{\tilde{k}/k}{\mathfrak{a}}\right) \in G(\tilde{k}/k)$$

through the Artin map.

The next lemma and corollary permit us to handle the capitulation problem in k by computation in the Galois group $G(\tilde{k}/k)$.

Lemma 1.1. *We have $A(k) = B(k)$ if and only if the restriction map $\varphi \mid B(k) : B(k) \to G(\tilde{k}/k)$ is surjective.*

Proof. Let $\bar{G} = G(\bar{k}/\boldsymbol{Q})$, $X = G(\bar{k}/k)$ and $G = G(k/\boldsymbol{Q}) = \langle\sigma\rangle$. Then G acts on X by an inner automorphism. Since at least one prime ideal is totally ramified in $k/\boldsymbol{Q}$, the group extension $1 \to X \to \bar{G} \to G \to 1$ splits. Hence we see that $[\bar{G}, \bar{G}] = X^{\sigma-1}$ and so $A(k)/A(k)^{\sigma-1} \simeq X/X^{\sigma-1} \simeq G(\tilde{k}/k)$ as G-module. Assume that $\varphi \mid B(k)$ is surjective. Then $A(k) = B(k)A(k)^{\sigma-1}$. Since the order of σ is ℓ and the order of $A(k)$ is a power of ℓ, this implies $A(k) = B(k)$. The converse is trivial.

Corollary 1.2. *Assume that $\varphi \mid B(k)$ is surjective. Then an ideal $\mathfrak{a}$ of k whose class belongs to $A(k)$ is principal if and only if* $\left(\frac{\tilde{k}/k}{\mathfrak{a}}\right) = 1$.

Proof. We have $\bar{k} = \tilde{k}$ because $A(k) = B(k)$.

2. Results. For a prime number p congruent to one modulo ℓ, we denote by k_p the unique subfield of $\boldsymbol{Q}(\zeta_p)$ of degree ℓ, where ζ_p is a primitive p-th root of unity. Let q be another prime number congruent to one modulo ℓ. Then $k_p k_q$ is an (ℓ, ℓ)-extension of $\boldsymbol{Q}$ and has $\ell - 1$ subfields which are cyclic extensions of $\boldsymbol{Q}$ of degree ℓ, in which both p and q are ramified. Let k be one of such subfields and $\mathfrak{p}_p$ (resp. $\mathfrak{p}_q$) the prime ideal of k lying over p (resp. q). Then $B(k) = \langle \mathrm{cl}(\mathfrak{p}_p), \mathrm{cl}(\mathfrak{p}_q)\rangle$ and $|B(k)| = \ell$. Note that $k_p k_q$ is the ℓ-part of the genus field of $k/\boldsymbol{Q}$. Since $\mathfrak{p}_p$ is ramified in $k/\boldsymbol{Q}$, $\left(\frac{k_p k_q/k}{\mathfrak{p}_p}\right)$ is trivial if and only if $\left(\frac{k_q/\boldsymbol{Q}}{p}\right)$ is trivial. Let $\left(\frac{p}{q}\right)_\ell$ denote the ℓ-th power residue symbol. Then the following lemma is an immediate consequence of Lemma 1.1.

Lemma 2.1. *We have $|A(k)| = \ell$ if and*

1991 Mathematics Subject Classification, Primary 11R23.