On Classification of Elliptic Fibrations with Small Number of Singular Fibres Over a Base of Genus 0 and 1

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Fix an algebraically closed field k of characteristic $p \neq 2,3$. Let $f: X \to C$ be a non-trivial Jacobian elliptic fibration defined over k with a base, a smooth projective curve C. In our consideration we always assume that f is relatively minimal and "Jacobian" means that f has a global section. Recall Shioda's formula as a special case of the Ogg-Shafarevich formula (cf. [6]).

(1) $\quad r + \rho_2 = 4g(C) - 4 + 2s - s_1$

where ρ_2 denotes the so-called Lefschetz number (the difference between the 2-nd Betti number b_2 and the Picard number ρ), r is the Mordell-Weil rank, s the number of singular fibres and s_1 denotes the number of semi-stable singular fibres (*i.e.* of type I_n in the Kodaira-Néron classification).

Since $\rho_2 \geq 0$ (Igusa's inequality), so if $C \simeq \boldsymbol{P}^1$ then from (1) it is clear that $s \geq 2$. On the other hand a non-trivial elliptic fibration over any base must have at least one singular fibres, because the moduli space of elliptic curves defined over k is $\boldsymbol{A}^1_k$. It is known also that the case $s = 1$ over an elliptic base is in fact realized. Note one more fact: if $C \simeq \boldsymbol{P}^1$ and f is non-isotrivial then $s \geq 3$. This fact should be thought in a different context and in a more general situation (cf. [3]). From the classification below one obtains another proof of this fact: in other words, one sees that elliptic fibrations over $\boldsymbol{P}^1$ with $s = 2$ are isotrivial.

Theorem 1. In the situation above assume that $K\text{-}S(f) \neq 0$. Then we have:

A. In the case $C \simeq \boldsymbol{P}^1$ and $s \leq 3$: X is a rational or $K3$ surface. Furthermore one has the following complete list (for completeness isotrivial fibrations are also included).

1. Rational surfaces ($s = 2$, $r = 0$):
$X_{22}(II^*, II)$, $X_{33}(III^*, III)$, $X_{44}(IV^*, IV)$, $X_{11}(j)(I_0^*, I_0^*)$ with $j \in k$.

2. Rational surfaces ($s = 3$)
 1) ($r = 0$): $X_{141}(I_1^*, I_4, I_1)$, $X_{222}(I_2^*, I_2, I_2)$, $X_{431}(IV^*, I_3, I_1)$, $X_{411}(I_4^*, I_1, I_1)$, $X_{321}(III^*, I_2, I_1)$, $X_{211}(II^*, I_1, I_1)$;
 2) ($r = 1$): $X^2_{321}(I_2^*, III, I_1)$, $X^3_{321}(I_1^*, III, I_2)$, $X^1_{211}(I_1^*, IV, I_1)$, $X^1_{341}(III^*, II, I_1)$, $X^2_{341}(IV^*, III, I_1)$, $X^2_{431}(I_3^*, II, I_1)$, $X^3_{431}(I_1^*, I_3, II)$, $X^1_{442}(IV^*, I_2, II)$;
 3) ($r = 2$): $X_{444}(IV, IV, IV)$, $X^1_{33}(I_0^*, III, III)$, $X^3_{341}(I_1^*, III, II)$, $X^2_{442}(I_2^*, II, II)$, $X^1_{11}(0)(I_0^*, IV, II)$, $X^1_{444}(IV^*, II, II)$.

3. $K3$ surfaces ($s = 3$):
$X^*_{411}(I_4^*, I_1^*, I_1^*)$, $X^*_{222}(I_2^*, I_2^*, I_2^*)$, $X^*_{431}(I_3^*, IV^*, I_1^*)$, $X^*_{321}(III^*, I_2^*, I_1^*)$, $X^*_{211}(II^*, I_1^*, I_1^*)$, $X^*_{11}(0)(II^*, IV^*, I_0^*)$, $X^*_{33}(III^*, III^*, I_0^*)$, $X^*_{341*}(III^*, IV^*, I_1^*)$, $X^*_{444}(II^*, II^*, IV)$, $X_{442*}(IV^*, IV^*, I_2^*)$, $X_{444*}(IV^*, IV^*, IV^*)$.

Moreover these surfaces are unique.

B. In the case $C \simeq E$, an elliptic curve, and $s = 1$, the fibration $f: X \to E$ has a unique configuration (I_6^*).

In characteristic zero, formula (1) is sufficient to conclude: $p_g(X) \leq 1$. In the general case it requires involving the so-called function field analog of Szpiro's conjecture which we formulate below.

Theorem ([1, Theorem 3]). Let $f: X \to C$ be a non-isotrivial family of elliptic curves (*i.e.* j-invariant is non-constant) with conductor of degree m. Then

(2) $\quad \deg(\Delta) \leq 6p^e(2g(C) - 2 + m)$

where Δ is the discriminant divisor on C and e is the inseparability exponent of the induced j-map: $C \to \boldsymbol{P}^1$.

First of all we remark that isotrivial case

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