A Remark on Boston's Question Concerning the Existence of Unramified p-Extensions. II

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§1. Introduction. Fontaine and Mazur have conjectured that there does not exist an everywhere unramified p-adic representation of the absolute Galois group of a number field with infinite image.

Conjecture 1 (Fontaine-Mazur). If K is a number field, and $\rho : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(\boldsymbol{Q}_p)$ an everywhere unramified representation, then the image of ρ is finite.

This conjecture has been studied in [1], [2] and [3].

Definition. A pro-p group G is called powerful if $G/\overline{G^p}$ (resp. $G/\overline{G^4}$) is abelian for p odd (resp. $p = 2$), where the line denotes topological closure.

Conjecture 1 is equivalent to the following (cf. [1]).

Conjecture 2 (Fontaine-Mazur-Boston). If K is a number field and M/K is an unramified pro-p extension of infinite degree, then the Galois group $\mathrm{Gal}(M/K)$ is not powerful.

In [1], Boston pointed out that this conjecture is closely related to the existence of unramified p-extensions of a certain type, and introduced the following question.

Question (Boston [1]). Let K be a number field, p an odd prime, and $K(p)$ its p-class field. Suppose that the class number of $K(p)$ is divisible by p. Then is there always an everywhere unramified extension M of degree p of $K(p)$ such that M is Galois over K and $exp(\mathrm{Gal}(M/K)) = exp(\mathrm{Gal}(K(p)/K))$? The "*exp*" stands for the exponent of the group.

In general, the answer to this question is in the negative. A counter example noted by Lemmermeyer can be found in Boston [2].

Concerning this question, Boston [1] noticed that the truth of the Fontaine-Mazur conjecture implies an affirmative answer, when K has an infinite p-class field tower. In the previous paper [5], we proved some sufficient conditions for the answer to Boston's question for K and p to be affirmative. In this article, we shall prove another sufficient condition for the answer to the question to be affirmative, and study the structure of $\mathrm{Gal}(K^{ur}(p)/K)$, where $K^{ur}(p)/K$ is the maximal unramified pro-p extension.

§2 Main theorem. Let k be an algebraic number field and p an odd prime. For a Galois extension L/k, we denote by $Ram(L/k)$ the set of primes of k which are ramified in L/k. For a finite set S of primes of k, let $B_k(S) = \{\alpha \in k^* \mid (\alpha) = \mathfrak{a}^p$ for some ideal $\mathfrak{a}$ of k, and $\alpha \in {k_{\mathfrak{q}}}^p$ for any $\mathfrak{q}$ of $S\}$.

Theorem. Assume that the Galois extension F/k satisfies the following conditions (1), (2), and (3).

(1) $\mathrm{Gal}(F/k)$ is isomorphic to $\boldsymbol{Z}/p\boldsymbol{Z} \times \boldsymbol{Z}/p\boldsymbol{Z}$.

(2) Any prime of k above p is unramified, and any prime contained in $Ram(F/k)$ is decomposed in F/k.

(3) $B_k(Ram(F/k)) = k^{*p}$.

If K/k is a p-extension such that $F \cap K = k$ and that $Ram(F/k) \subset Ram(K/k)$, then the answer to Boston's question for K and p is affirmative.

We need the lemma below.

Lemma ([4; Corollary of Theorem 4]). Let F/k be a Galois extension with the Galois group isomorphic to $\boldsymbol{Z}/p\boldsymbol{Z} \times \boldsymbol{Z}/p\boldsymbol{Z}$. Assume that $B_k(Ram(F/k)) = k^{*p}$. Then the following condisions (1) and (2) are equivalent.

(1) There exists a Galois extension $L/F/k$ such that $\mathrm{Gal}(L/k)$ is isomorphic to

$$H = \langle x, y \mid x^p = y^p = z^p = 1, \quad yx = xyz, xz = zx, yz = zy \rangle$$

and that L/F is unramified.

(2) Any prime of k which is ramified in F/k is decomposed in F/k.

Remark. H is a non-abelian p-group of order p^3, and the exponent of H is p. Therefore