Regular and Stable Points in Dirichlet Problem

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(Communicated by Kiyosi ITÔ, M. J. A., Jan. 13, 1997)

Consider a subregion M of *Carathéodory type* of the extended Euclidean space $\bar{\boldsymbol{R}}^d = \boldsymbol{R}^d \cup \{\infty\}$ of dimension $d \geq 2$, i.e. M is a subregion of $\bar{\boldsymbol{R}}^d$ such that the boundary ∂M of M is contained in $\boldsymbol{R}^d$ and $\partial \bar{M} = \partial M$. A sequence $(M_i)_{i \geq 1}$ of subregions M_i of $\bar{\boldsymbol{R}}^d$ is referred to as a *squeezer* of $\bar{M}$ if $M_i \supset M_{i+1} \supset \bar{M}$ for every $i \geq 1$ and $\cap_{i \geq 1} M_i = \bar{M}$. For any $f \in C(\bar{\boldsymbol{R}}^d)$ we denote by H_f^M the harmonic Dirichlet solution for the boundary function $f \mid \partial M$ on M obtained by the Perron-Wiener-Brelot method (cf. e.g. [4]). It is known as the Wiener type theorem that the sequence $(H_f^{M_i})_{i \geq 1}$ converges pointwise on $\bar{M}$ and locally uniformly on M for any $f \in C(\bar{\boldsymbol{R}}^d)$ and for any squeezer $(M_i)_{i \geq 1}$ of $\bar{M}$. It is convenient to introduce the notation

$$H_f^{\bar{M}}(x) := \lim_{i \to \infty} H_f^{M_i}(x) \quad (x \in \bar{M})$$

which is harmonic on M and depends only on $f \mid \partial M$ and $\bar{M}$ independent of the choice of the squeezer $(M_i)_{i \geq 1}$. The function $H_f^{\bar{M}}$ is sometimes referred to as the external solution of the Dirichlet problem for the domain M with the boundary function f and also given by

$$(1) \qquad H_f^{\bar{M}}(x) = \int_{\partial M} f(y)\, d\beta_{\bar{M}^c}\varepsilon_x(y),$$

where ε_x is the Dirac measure with its support at x and $\beta_{\bar{M}^c}$ denotes the balayage operation for the set $\bar{M}^c$ (cf. [6, §5 in Chap. V]). The Dirichlet problem is said to be *stable inside* M (*stable in* $\bar{M}$, resp.) if $H_f^{\bar{M}} = H_f^M$ on M (if $(H_f^{M_i})_{i \geq 1}$ converges uniformly to f on ∂M, resp.). The stability in $\bar{M}$ implies the stability inside M. In particular, the stability in $\bar{M}$ is closely related to the harmonic approximation question (cf. e.g. [6], [3], [1], etc.). To judge the stability for concrete regions it is convenient to localize the stability. In his celebrated paper [5] Keldysh introduced the following notion: a boundary point $y \in \partial M$ is said to be a *stable point* if $H_f^{\bar{M}}(y) = f(y)$ for every $f \in C(\bar{\boldsymbol{R}}^d)$. A point $y \in \partial M$ which is not a stable point is termed as an *unstable point*. In view of (1) it is readily seen that $y \in \partial M$ is a stable point if and only if y is a regular point of the set $\bar{M}^c$ in the sense of [6, Chap. V].

In terms of stable points Keldysh [5] showed the following: the Dirichlet problem is stable inside M if and only if the set of all unstable points in ∂M is of harmonic measure zero relative to M ; the Dirichlet problem is stable in $\bar{M}$ if and only if every boundary point in ∂M is stable. As for the relation of stability of boundary points to the regularity (cf. e.g. [4]) of them, Keldysh [5] proved that a stable boundary point $y \in \partial M$ is automatically a regular boundary point for the Dirichlet problem on M but there is an example (i.e. the so called Keldysh ball (cf. no. 12 below)) indicating that the converse of the above is not true. There are many handy geometric criterion for the regularity and therefore it will be usefull to give a practical geometric condition under which the regularity implies the stability for boundary points. The *purpose* of this paper is to give such an easily applicable condition. Roughly speaking (cf. no. 3 below for precise definition), a boundary point $y \in \partial M$ is said to be *graphic* if one of the following two conditions is satisfied: there exist a neighborhood U of y, a Cartesian coordinate $x = (x^1, \cdots, x^{d-1}, x^d) = (x', x^d)$, and a continuous function $\psi(x')$ of x' such that $(\partial M) \cap U$ is represented as the graph of the function $x^d = \psi(x')$ and $M \cap U$ is situated on only one side of the graph; there exist a neighborhood U of y, a polar coordinate (r, ξ) $(r \geq 0, |\xi| = 1)$, and a continuous function $\phi(\xi) \geq 0$ of ξ such that $(\partial M) \cap U$ is represented as the graph of the function $r = \phi(\xi)$ and $M \cap U$ is situated on only one side of the graph.

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1991 *Mathematics Subject Classification*. Primary 31B25; Secondary 31B05, 31B20.

This work was partly supported by Grant-in-Aid for Scientific Research, No. 07640196, Ministry of Education, Science, Sports and Culture, Japan.