On the Problems of Conformal Maps with Quasiconformal Extension

By HUANG Xinzhong*) and Shigeyoshi OWA**)

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1. Introduction. Let $f(z)$ be meromorphic and locally univalent in the unit disk $\boldsymbol{D} = \{z : |z| < 1\}$. Then the Schwarzian derivative of $f(z)$ is defined as

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

It is well-known that if $f(z)$ is locally univalent in $\boldsymbol{D}$ and satisfies

$$|S_f(z)| \leqq \frac{2}{(1-|z|^2)^2} \quad (z \in \boldsymbol{D}),$$

then $f(z)$ is univalent in $\boldsymbol{D}$. Furthermore, if

$$(1) \qquad |S_f(z)| \leqq \frac{2t}{(1-|z|^2)^2} \quad (z \in \boldsymbol{D})$$

for some $t (0 \leqq t < 1)$, then $f(z)$ has a quasiconformal extension to the plane.

Chuaqui and Osgood [2] have proved that

Theorem A. Let $f(z)$ be analytic in $\boldsymbol{D}$ with $f(0) = 0, f'(0) = 1$, and $f''(0) = 0$. If $f(z)$ satisfies (1) then

$$A(|z|, -t) \leqq |f(z)| \leqq A(|z|, t)$$

and

$$A'(|z|, -t) \leqq |f'(z)| \leqq A'(|z|, t)$$

for $z \in \boldsymbol{D}$, where A' means the differentiation of A with respect to z, and $A(z, t)$ is defined as

$$(2)\ A(z, t) = \left(\frac{1}{\sqrt{1-t}}\right)\frac{(1+z)^{\sqrt{1-t}} - (1-z)^{\sqrt{1-t}}}{(1+z)^{\sqrt{1-t}} + (1-z)^{\sqrt{1-t}}}.$$

Using Theorem A, they also proved that

Theorem B. If $f(z)$ which is normalized as in Theorem A is analytic in $\boldsymbol{D}$, and satisfies (1), then $f(z)$ has a Hölder continuous extension to $|z| \leqq 1$ with

$$|f(z_1) - f(z_2)| \leqq \frac{4\pi}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}},$$

for all z_1 and z_2 in $\boldsymbol{D}$. The exponent $\sqrt{1-t}$ is sharp.

In Theorem B, although the exponent $\sqrt{1-t}$ is sharp, the Hölder constant $4\pi/\sqrt{1-t}$ is not sharp.

2. Hölder continuous extension. Our first result on Hölder continuous extension is contained in

Theorem 1. Let $f(z)$ be analytic in $\boldsymbol{D}$ with $f(0) = 0, f'(0) = 1$, and $f''(0) = 0$. If $f(z)$ satisfies (1), then $f(z)$ has a Hölder continuous extension to $|z| \leqq 1$ with

$$|f(z_1) - f(z_2)| \leqq \left(\frac{4}{1-\sqrt{1-t}}\right)^{1-\sqrt{1-t}} \frac{1-\sqrt{1-t} + 2^{1-\sqrt{1-t}}\sqrt{1-t}}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}}$$

for all z_1 and z_2 in $|z| \leqq 1$. The exponent $\sqrt{1-t}$ is sharp.

Proof. According to Chuaqui and Osgood [2], we have

$$(3) \qquad |f'(z)| \leqq 4\frac{(1+|z|)^{2\nu-1}(1-|z|)^{2\nu-1}}{((1+|z|)^{2\nu} + (1-|z|)^{2\nu})^2} \quad (2\nu = \sqrt{1-t}),$$

and

$$(4) \qquad |f'(z)| \leqq \frac{4^{1-2\nu}}{(1-|z|)^{1-2\nu}}$$

for $z \in \boldsymbol{D}$. Let z_1 and $z_2 (z_1 \neq z_2)$ be arbitrary points in $\boldsymbol{D}$ and choose $\rho = 1 - (1-2\nu)|z_1 - z_2|/2$. Then, from (4), we have

$$\begin{aligned}
|f(z_1) - f(z_2)| &\leqq \left|\int_{z_1}^{\rho z_1} f'(z)dz\right| + \left|\int_{\rho z_1}^{\rho z_2} f'(z)dz\right| + \left|\int_{\rho z_2}^{z_2} f'(z)dz\right| \\
&\leqq 2\int_\rho^1 \frac{4^{1-2\nu}}{(1-r)^{1-2\nu}}dr + \frac{4^{1-2\nu}}{(1-\rho)^{1-2\nu}}|z_1 - z_2| \\
&= \frac{4^{1-\sqrt{1-t}}}{\sqrt{1-t}}\left(\frac{1-\sqrt{1-t}}{2}\right)^{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}} + \frac{4^{1-\sqrt{1-t}}\,2^{1-\sqrt{1-t}}}{(1-\sqrt{1-t})^{1-\sqrt{1-t}}}|z_1 - z_2|^{\sqrt{1-t}} \\
&= \frac{1}{\sqrt{1-t}}\left(\frac{8}{1-\sqrt{1-t}}\right)^{1-\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}} \\
&\leqq \frac{8}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}} \\
&< \frac{4\pi}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}}.
\end{aligned}$$

This gives a better result than Theorem B.

*) Department of Mathematics, Huaqiao University, China.

**) Department of Mathematics, Kinki University.