Connection Formulae for Solutions of a System of Partial Differential Equations Associated with the Confluent Hypergeometric Function Φ_2

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1. Introduction. Consider the confluent hypergeometric function

(1) $\Phi_2(\beta, \beta', \gamma, x, y) = \sum_{m,n\geq 0} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}(1)_m(1)_n} x^m y^n$

convergent for $|x| < \infty$, $|y| < \infty$, in which $(\beta)_m = \Gamma(\beta + m)/\Gamma(\beta)$ (cf. [3]). This function satisfies a system of partial differential equations

(2) $$xz_{xx} + yz_{xy} + (\gamma - x)z_x - \beta z = 0,$$
$$yz_{yy} + xz_{xy} + (\gamma - y)z_y - \beta' z = 0,$$

which possesses the singular loci $x = 0$, $y = 0$, $x - y = 0$ of regular type and $x = \infty$, $y = \infty$ of irregular type. The solutions of system (2) constitute a three-dimensional vector space over $\boldsymbol{C}$. In what follows, we assume that none of the complex numbers $\beta, \beta', \gamma - \beta - \beta', \beta - \gamma, \beta' - \gamma$, and $\beta + \beta'$ is an integer, and use the notation $e^{(\lambda)} = \exp(2\pi i\lambda)$.

It is known by Erdélyi [1,2] that, near the singular loci of irregular type, system (2) admits convergent solutions as follows:

$$u_0 = \Phi_2(\beta, \beta', \gamma, x, y) \quad (|x| < \infty, |y| < \infty),$$

$$v_1 = x^{\beta'-\gamma+1}y^{-\beta'}\Phi_1(\beta + \beta' - \gamma + 1, \beta', \beta' - \gamma + 2, x/y, x) \quad (|x| < |y|)$$
$$= x^{\beta'-\gamma+1}(y - x)^{-\beta'} \times \Phi_1(1 - \beta, \beta', \beta' - \gamma + 2, x/(x - y), -x) \quad (|x| < |x - y|),$$

$$v_2 = x^{-\beta}y^{\beta-\gamma+1} \times \Phi_1(\beta + \beta' - \gamma + 1, \beta, \beta - \gamma + 2, y/x, y) \quad (|y| < |x|),$$

$$v_3 = x^{\beta+\beta'-\gamma}(y - x)^{1-\beta-\beta'}e^x\Phi_1(1 - \beta, \gamma - \beta - \beta', 2 - \beta - \beta', (x - y)/x, y - x) \quad (|x - y| < |x|),$$

$$w_1 = y^{1-\gamma}\Gamma_1(\beta, \beta' - \gamma + 1, \gamma - 1, -x/y, -y) \quad (|x| < |y|)$$
$$= (y - x)^{1-\gamma}e^x\Gamma_1(\gamma - \beta - \beta', \beta' - \gamma + 1, \gamma - 1, x/(y - x), x - y) \quad (|x| < |x - y|),$$

$$w_2 = x^{1-\gamma}\Gamma_1(\beta', \beta - \gamma + 1, \gamma - 1, -y/x, -x) \quad (|y| < |x|),$$

$$w_3 = x^{1-\gamma}e^x \times \Gamma_1(\beta', 1 - \beta - \beta', \gamma - 1, (y - x)/x, x) \quad (|x - y| < |x|),$$

where

$$\Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m,n\geq 0} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}(1)_m(1)_n} x^m y^n,$$

$$\Gamma_1(\alpha, \beta, \beta', x, y) = \sum_{m,n\geq 0} \frac{(\alpha)_m(\beta)_{n-m}(\beta')_{m-n}}{(1)_m(1)_n} x^m y^n$$

are convergent for $|x| < 1$, $|y| < \infty$. Hence we have triplets of linearly independent solutions (u_0, v_1, w_1) (in the domain $|x| < |y|$ or $|x| < |x - y|$), (u_0, v_2, w_2) (in the domain $|y| < |x|$) and (u_0, v_3, w_3) (in the domain $|x - y| < |x|$).

On the other hand, in [4,5], we chose linearly independent solutions expressed as

(3) $$z_+ = (1 - e^{(\beta)})^{-1} \int_{C(x)} f(x, y, t)dt,$$

(4) $$z_0 = (1 - e^{(\gamma-\beta-\beta')})^{-1} \int_{C(0)} f(x, y, t)dt,$$

(5) $$z_- = (1 - e^{(\beta')})^{-1} \int_{C(y)} f(x, y, t)dt,$$

with

(6) $f(x, y, t) = t^{\beta+\beta'-\gamma}(t - x)^{-\beta}(t - y)^{-\beta'}e^t$,

and examined the asymptotic behaviour of them near the singular loci $x = \infty$, $y = \infty$ of irregular type. Here the paths of integration and the branch of the integrand are taken in such a way that, in the case where

(7) $$0 < \arg x < \pi < \arg y < 2\pi,$$
$$\pi < \arg(y - x) < 2\pi,$$

they have the following properties:

(i) $C(a)$ $(a = 0, x, y)$ is a loop which starts from $t = -\infty$, encircles $t = a$ in the positive sense, and ends at $t = -\infty$.

(ii) $C(x)$ lies over $C(0)$, and $C(y)$ lies under $C(0)$ in the t-plane.

(iii) The branch of $f(x, y, t)$ is taken such that $\arg t = \arg(t - x) = \arg(t - y) = \pi$ at the end point $t = -\infty$ of each path of integration.

In this paper, we calculate connection