The Relative Class Number of Certain Imaginary Abelian Number Fields of Odd Conductors*)

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(Communicated by Shokichi IYANAGA, M. J. A., March 12, 1996)

1. Introduction. The class number of an imaginary abelian number field is divisible by that of its maximal real subfield and the quotient is called the relative class number of it.

Let p be an odd prime number. For a rational integer a prime to p, we denote by $R(a)$ the least positive residue of a modulo p. Then Maillet's determinant D_p is defined by

$$D_p = | R(ab') |_{1 \leq a,b \leq r},$$

where $r = (p-1)/2$ and b' is a rational integer which satisfies $bb' \equiv 1 \pmod p$.

Let Q and ζ be the field of rational numbers and a primitive p-th root of unity, respectively. Carlitz and Olson [1] proved that D_p is a multiple of the relative class number h_p^- of the p-th cyclotomic number field $Q(\zeta)$. This result has been generalized to more general imaginary abelian number fields [8], [11], [12], [15], [16].

On the other hand, recently Hazama [10] showed that the determinant of the Demjanenko matrix provides the formula for h_p^-. The Demjanenko matrix is defined by

$$(C(ab))_{1 \leq a,b \leq r};$$

herein for a rational integer a prime to p $C(a) = 1$ if $R(a) < p/2$, and $C(a) = 0$ if $R(a) > p/2$. Hazama's formula has been also generalized to more general imaginary abelian number fields of odd conductors [2], [7], [9], [13].

In the previous papers [3], [4] we investigated the Stickelberger ideal of quadratic extensions of $Q(\zeta)$ and obtained a formula for the relative class number of such imaginary abelian number fields. Our formula is expressed as a product of two determinants of degree r. In this paper we consider the Demjanenko matrix and show a new relative class number formula expressed as a product of two determinants of degree r.

*) Dedicated to Professor Katsumi Shiratani on his 63rd birthday.

2. Statement of the theorem. Let m be a square-free rational integer such that $m \equiv 1 \pmod 4$, and d its absolute value. We consider the quadratic extension $K = Q(\zeta, \sqrt{m})$ of $Q(\zeta)$ obtained by adjoing $\sqrt{m}$. We may assume without loss of generality that m is prime to p. Let Z be the ring of rational integers and N the subgroup of the multiplicative group $(Z/dZ)^\times$ corresponding to $Q(\sqrt{m})$ by Galois theory; then the Galois group G of K/Q is isomorphic to the direct product of the multiplicative group $(Z/pZ)^\times$ and the quotient group $(Z/dZ)^\times/N$.

For each $1 \leq a \leq p-1$ we choose a rational integer a^* prime to dp so that $a^* \equiv a \pmod p$ and $1^*, 2^*, \ldots, (p-1)^*$ form a complete system of representatives for $G/\{\pm 1\}$; then we see $(p-a)^* \not\equiv -a^* \pmod N$ and we may take $1^* = 1$.

Now, for a rational integer a prime to dp we denote by $c_a^{(K)}$ and $c_a'^{(K)}$ respectively the number of $1 \leq x \leq (dp-1)/2$ such that $x \equiv a \pmod p$ and $x \equiv a \pmod N$, and that of $(dp+1)/2 \leq x \leq dp-1$ such that $x \equiv a \pmod p$ and $x \equiv a \pmod N$. We define the Demjanenko matrix for K by

$$(c_{a^*b^*}^{(K)} - c_{b^*}'^{(K)})_{1 \leq a,b \leq p-1}$$

[2], and denote its determinant by $H^{(K)}$.

Let X be the group of the primitive Dirichlet characters associated with $Q(\sqrt{m})$, and further $\chi_0 \in X$ the principal character of conductor d. For any $\chi \in X$ and a rational integer a prime to p, let

$$C_a(\chi) = \sum_{x=1}^{(dp-1)/2}{}^{(a)} \chi(x)$$

and

$$C_a'(\chi) = \sum_{x=(dp+1)/2}^{dp-1}{}^{(a)} \chi(x),$$

where (a) indicates that x runs through rational integers in the assigned interval which are prime to dp and congruent to a modulo p. We then define a determinant $H_p(\chi)$ of degree r by