Discretization of Non-Lipschitz Continuous O.D.E. and Chaos

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1. Introduction. In the previous paper [1], we investigated Euler's discretization of the scalar autonomous ordinary differential equation which has only one stable equilibrium point. Under some conditions, it is shown that Euler's finite difference scheme $F_{\Delta t}$ is chaotic for a sufficiently large fixed time step Δt.

On the contrary, in this paper, for a sufficiently small fixed time step Δt, we will find the necessary and sufficient conditions under which $F_{\Delta t}$ is stable in the neighborhood of the equilibrium point, and the sufficient conditions under which $F_{\Delta t}$ is chaotic around the equilibrium point.

2. Definitions and assumptions. For the scalar autonomous O.D.E.

$$(1) \qquad \frac{du}{dt} = f(u) \quad u \in \boldsymbol{R}^1,$$

we put following assumptions:

$$\begin{cases} f(u) \text{ is continuous in } \boldsymbol{R}^1 \\ f(u) > 0 \quad (u < 0) \\ f(0) = 0 \\ f(u) < 0 \quad (0 < u). \end{cases}$$

In other words, $u = 0$ is the only stable equilibrium point. Euler's discretization scheme for (1) is as follows: with the fixed time step Δt,

$$\frac{x_{n+1} - x_n}{\Delta t} = f(x_n),$$

$$x_{n+1} = x_n + \Delta t \cdot f(x_n).$$

Now, finite difference scheme $F_{\Delta t}(x)$ is defined as

(2) $F_{\Delta t}(x) = x + \Delta t \cdot f(x)$, (i.e. $x_{n+1} = F_{\Delta t}(x_n)$)

and we will investigate this dynamical system $F_{\Delta t}(x)$.

3. Condition for stable behavior of $F_{\Delta t}$. Generally speaking, Euler's finite difference scheme with sufficiently small Δt gives a good approximation for the solution of differential equation. For example, consider a differential equation

$$\frac{du}{dt} = au(1-u) \quad (u \geq 0,\ a \text{ is a positive constant}).$$

The orbits of the corresponding dynamical system (2) converge to a stable equilibrium point $u = 1$ with any Δt less than $2/a$. But the next example shows that however small Δt is chosen, the orbits don't always converge to the equilibrium point:

$$\frac{du}{dt} = \begin{cases} \sqrt{-u} & (u < 0) \\ -\sqrt{u} & (u \geq 0). \end{cases}$$

In this case, $F_{\Delta t}(x)$ is super-unstable at $x = 0$ ($F'_{\Delta t}(0) = -\infty$), and it has a super-stable orbit ($\pm \Delta t^2/4$) with period 2.

Theorem 1(Lipschitz case). Assume that (1) holds the following additional condition:

$$(3) \qquad \left| \frac{f(u)}{-u} \right| < M_0 \quad (\forall u < 0)$$

(M_0 is a positive constant).

Then, there exists $\Delta T > 0$, such that for any $\Delta t (0 < \Delta t < \Delta T)$, $F_{\Delta t}$ has no periodic orbit except the equilibrium point $x = 0$. And for any initial point x_0, $F^n_{\Delta t}(x_0)$ converges to the equilibrium point.

Proof of Theorem 1. Define subsets D_-, D_+, D_0 andD' of $\boldsymbol{R}^2$ by

$$D_- = \{(x, y) \mid x < y < 0\},$$
$$D_+ = \{(x, y) \mid 0 < y < x\}$$
$$D_0 = \{(x, y) \mid 0 < x,\ y = 0\},$$
$$D' = \{(x, y) \mid y < 0 < x\}.$$

Set $\Delta T = 1/M_0$. From the condition (3), for any $\Delta t (0 < \forall \Delta t < \Delta T)$ and any $x < 0$,

$$F_{\Delta t}(x) = x + \Delta t \cdot f(x) < x + \Delta T \cdot f(x) < x + \Delta T \cdot (-M_0 x) = x(1 - M_0 \Delta T) = 0.$$

On the other hand, $F_{\Delta t}(x) = x + \Delta t \cdot f(x) > x$, so $x < F_{\Delta t}(x) < 0$.

Hence, $x < 0$ implies $(x, F_{\Delta t}(x)) \in D_-$ for any $\Delta t (0 < \forall \Delta t < \Delta T)$.

Let $x_n = F^n_{\Delta t}(x_0)$ $(n \geq 0)$ be an orbit of $F_{\Delta t}$. There are 4 cases of behavior of x_n as follows:

Case (a) $x_0 < 0$. Then $(x_n, x_{n+1}) \in D_-$ for any $n \geq 0$. Therefore the sequence x_n increases monotonously towards the equilibrium point.

Case (b) $x_0 > 0$, and $(x_n, x_{n+1}) \in D_+$ for any $n \geq 0$. Then the sequence x_n decreases monotonously towards the equilibrium point.