Fundamental Solution of Anisotropic Elasticity

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(Communicated by Kiyosi ITÔ, M. J. A., Feb. 13, 1996)

Abstract: A formula for the fundamental solution of three-dimensional anisotropic elasticity is given in terms of the eigenvectors and/or the generalized eigenvectors of the associated six-dimensional eigenvalue problem called Stroh's eigenvalue problem. From this formula an explicit closed form of the fundamental solution for transversely isotropic media is obtained.

1. Introduction. The aim of the present paper is to give an explicit formula of the fundamental solution of three-dimensional anisotropic elasticity. Let $C = (C_{ijkl})_{1\le i,j,k,l\le 3}$ be a three-dimensional homogeneous linear anisotropic elastic tensor which satisfies the following symmetry and strong convexity conditions;

(A-1) $C_{ijkl} = C_{klij}$ $(1 \le i, j, k, l \le 3)$

(A-2) $\exists \delta > 0 ; \sum_{i,j,k,l=1}^{3} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \ge \delta \sum_{i,j=1}^{3} \varepsilon_{ij}^2$ for a real matrix $\mathscr{E} = (\varepsilon_{ij})$.

Let $x \in R^3$ and let $G_{km} = G_{km}(x)$ be a solution to

$$\sum_{j,k,l=1}^{3} C_{ijkl}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}G_{km} + \delta_{im}\delta(x) = 0 \text{ in } R^3 \quad (1 \le i, m \le 3)$$

where (x_1, x_2, x_3), δ_{im} and $\delta(x)$ are the cartesian coordinates of x, the Kronecker delta symbol and the Dirac delta function, respectively. $\boldsymbol{G} = \boldsymbol{G}(x) = (G_{km}\,;^{k\downarrow 1,2,3}_{m\to 1,2,3})$ is called the fundamental solution to the system of the equations of anisotropic elastostatics. Physically, the solution G_{km} describes the displacement at the point x in the x_k direction due to a point force at the origin in the x_m direction.

Bernett [1] gave a formula for $\boldsymbol{G}$ in terms of an integral on the interval $[0,2\pi]$ whose integrand was a smooth periodic function with a period 2π. Malén [5] gave another formula prior to [1]. That is he shows that $\boldsymbol{G}$ can be expressed in terms of the eigenvectors of Stroh's eigenvalue problem provided that all the eigenvalues are distinct. Malén's formula is useful for the perturbation argument for the fundamental solution (cf. Malén and Lothe [6], Nishioka and Lothe [9,10]) and the estimation of the displacement field and the stress field around a straight dislocation (cf. Malén [4,5]). However, the assumption of distinctness of the eigenvalues is too strict to hold for most crystals, since they have some symmetries.

In this paper we give a formula for $\boldsymbol{G}$ in terms of the eigenvectors and/or the generalized eigenvectors of Stroh's eigenvalue problem, which is slightly different from that of [5], without assuming distinctness of the eigenvalues. As a byproduct we give an explicit closed form of $\boldsymbol{G}$ for transversely isotropic media, because the explicit formulae of the eigenvectors and/or the generalized eigenvectors for the associated eigenvalue problem are available in the case of transversely isotropic media. The explicit closed form of the fundamental solution will be useful for computing the displacement and the stress fields in the elastic medium by the boundary element method, which is an effective method in numerical analysis derived from the integral equation methods for boundary value problems.

2. Result. Let $x \neq 0$. Write

$$\frac{x}{|x|} = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

in terms of the polar coordinates (r, φ, θ) $(r \ge 0, 0 \le \varphi \le \pi, 0 \le \theta < 2\pi)$. Let

$$\boldsymbol{v}^0 = (\sin\theta, -\cos\theta, 0),$$

$$\boldsymbol{w}^0 = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi).$$

Define $\boldsymbol{Q} = \boldsymbol{Q}(\varphi, \theta)$, $\boldsymbol{R} = \boldsymbol{R}(\varphi, \theta)$, $\boldsymbol{T} = \boldsymbol{T}(\varphi, \theta)$ by

$$\boldsymbol{Q} = \langle \boldsymbol{v}^0, \boldsymbol{v}^0\rangle, \boldsymbol{R} = \langle \boldsymbol{v}^0, \boldsymbol{w}^0\rangle, \boldsymbol{T} = \langle \boldsymbol{w}^0, \boldsymbol{w}^0\rangle,$$

where

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = (\langle \boldsymbol{v}, \boldsymbol{w}\rangle_{ik}\,;^{i\downarrow 1,2,3}_{k\to 1,2,3}),$$

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