On the Rank of the Elliptic Curve $y^2 = x^3 - 1513^2x$

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§1. Method to be used. Let $r_n$ be the rank of the elliptic curve $y^2 = x^3 - n^2x$. We will prove in this paper $r_n$ is two for $n = 1513$ using Tate’s method (cf. [3]).

If $x/y = u^2$ for some rational number $u$, we write $x = y^2$. Consider the diophantine equations

(1) $dX^4 - (n^2/d)Y^4 = Z^2$, $d | n^2$, $d + 1, d + n$

(2) $dX^4 + (4n^2/d)Y^4 = Z^2$, $d | 4n^2$, $d + 1$

Let $\{d_1, \ldots, d_v\}$ be the set of $d$’s for which (1) is solvable in $x, y, z$ with $(x, (n^2/d) YZ) = (y, dXZ) = 1$ and $(d_{i+1}, \ldots, d_{i+v})$ be the set of $d$’s for which (2) is solvable in $x, y, Z$ with $(X, (4n^2/d) YZ) = (Y, dXZ) = 1$ (we assume $d_i + d_j$ for $1 \leq i < j \leq \mu$ and for $\mu + 1 \leq i < j \leq \mu + v$). Then $2r^2 \geq (4 + \mu)(1 + \nu)$ which gives $r_n$.

For $n = 17 \cdot 89$, we have a solution of (1): $17^2 \cdot 89 \cdot 3^4 = 14242^2$ and a solution of (2): $2 \cdot 17 \cdot 89 \cdot 7^4 + 2 \cdot 17 \cdot 89 \cdot 5^4 = 3026^2$. Therefore we get $r_n \geq 2$. For proving $r_n = 2$, we must show that the next five diophantine equations have no solutions.

(3) $17 \cdot 89X^4 + 4 \cdot 17 \cdot 89 Y^4 = Z^2$

(4) $17X^4 + 4 \cdot 17 \cdot 89 Y^4 = Z^2$

(5) $17 \cdot 89X^4 + 4 \cdot 17Y^4 = Z^2$

(6) $89X^4 + 4 \cdot 17^2 \cdot 89 Y^4 = Z^2$

(7) $89 \cdot 17^2 X^4 + 4 \cdot 89 Y^4 = Z^2$

§2. Non solvability of (3)-(7). If (3) is solvable then $Z = 17 \cdot 89 W$ for some integer $W$ and we get $X^4 + 4Y^4 = 17 \cdot 89 W^2$. This equation can be written as $(X^2)^2 + (Y^2)^2 = (27^4 + 28^4)W^2$. We need next lemma (cf. [2] p. 317).

Lemma. When $a = \text{odd}$, $b = \text{even}$, $c = a^2 + b^2 = \text{square free}$, $(x, y) = 1$, $x = \text{odd}$. $y = \text{even}$ and $x^2 + y^2 = cz^2 = (a^2 + b^2)z^2$. Then we have

$(ax + by + cz)(ax - by - cz) = -c(y + bz)^2$

$d = (ax + by + cz, ax - by - cz) = \text{twice a square}$

Proof. Put $A = ax + by + cz$, $B = ax - by - cz$. Then

$AB = a^2 x^2 - b^2 y^2 - 2bcyz - c^2z^2$

$d = a^2(c^2 - y^2) - b^2 y^2 - 2bcyz - c^2z^2$

$= (c^2 - y^2 - 2bcyz - c^2z^2)$

As $A \equiv B \equiv 0 (\text{mod } 2)$ and $d | A + B = 2ax$, Then $p | \text{odd prime divisor of } d$. Then $p | ax$ and $p | y + bz$ because $c$ is square free. If $p | a$ then $p | (y + bz)(y - bz) = a^2 - x^2$. So we have $p | x$. If $p | x$ then $p | az$. But $(x, z) = 1$, so we have $p | a$. If $p | y - bz$ then $p | (y + bz) + (y - bz) = 2y$. But $(x, y) = 1$, so we have $p | x - bz$. Let $p^{2k} \parallel d$. When $k < l$ then $p^{2k} \parallel d$. When $k = l$, we have $p^{2k} \parallel d$. But $d | A + B = 2ax$, so we have $p^{2k} \parallel d$. Therefore $d$ is twice a square.

From this lemma, we can find $c_1, c_2, u, v$ such that

$ax = c_1u^2 - c_2v^2, c_1c_2 = c, 2uv = y + bz$

When $x = X^2$, $y = 2Y^2$, $z = W$, $a = 27, b = 28$ then $x = \text{odd}$ because of $(X, 4 \cdot 17 \cdot 89 YZ) = 1$ and we have

$27X^2 = c_1u^2 - c_2v^2, c_1c_2 = 17 \cdot 89$

Using $17 \equiv 1 \pmod{4}$, $(27 \pmod{17}) = -1, (89 \pmod{17}) = 1$, we have a contradiction. So (3) has no solution.

If (4) is solvable, then $Z = 17 \cdot 89 W$ for some integer $W$ and we get

$(X^2)^2 + (2 \cdot 89 Y^2)^2 = (1^2 + 4^2) W^2$

As $X$ is odd, we have $W = \text{odd}$. $Y = \text{even}$ and $X^2 = c_1u^2 - c_2v^2, c_1c_2 = 17$

$2uv = 2 \cdot 89 Y^2 + 4W \equiv 4 \pmod{8}$

From this we have $c_1u^2 - c_2v^2 \equiv \pm 3 \pmod{8}$. This is a contradiction. So (4) had no solution. In the same way, (5) has no solution.

If (6) is solvable, then $Z = 89W$ for some integer $W$ and we get

$(X^2)^2 + (2 \cdot 17 Y^2)^2 = (5^2 + 8^2) W^2$

As $X$ is odd, we have $W = \text{odd}, Y = \text{even}$ and $5X^2 = c_1u^2 - c_2v^2, c_1c_2 = 89$

$2uv = 2 \cdot 17 Y^2 + 8W \equiv 0 \pmod{8}$

Therefore $c_1u^2 - c_2v^2 \equiv \pm 1 \pmod{8}$. This is a