Triangles and Elliptic Curves. VII

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This is a continuation of series of papers [2] each of which will be referred to as (I), (II), (III), (IV), (V), (VI) in this paper. In (VI) we considered exclusively real triangles $t = (a, b, c)$ and showed that there is a 1-1 correspondence between the classes of similarity of t's and the isomorphism classes of triples $\boldsymbol{E}_t$'s of elliptic curves. In this paper, we pursue the same theme for those objects rational over any subfield k of $\boldsymbol{R}$. This time, we shall introduce a third object (a quartic surface over $\boldsymbol{Q}$) in addition to triangles and elliptic curves to clarify the matter.

§1. Tr and S_+. As in (VI), we begin with the set

(1.1) $Tr = \{t = (a, b, c) \in \boldsymbol{R}^3 ; 0 < a < b + c, \; 0 < b < c + a, \; 0 < c < a + b\}.$

For $t, t' \in Tr$, we write $t \sim t'$ if they are similar, i.e., if $t = rt'$ for some $r \in \boldsymbol{R}$. For any subfield $k \subset \boldsymbol{R}$, put

(1.2) $$Tr(k) = Tr \cap k^3.$$

If $t \sim t'$, $t, t' \in Tr(k)$, note that $t = rt'$ with $r \in k$. So we can speak of the embedding $\widetilde{Tr}(k) \subset \widetilde{Tr}$ of quotients in the obvious way.

Next, we consider the set

(1.3) $S_+ = \{P = (x, y, z) \in \boldsymbol{R}^3 ; x, y, z > 0, \; (xy)^{\frac{1}{2}} + (yz)^{\frac{1}{2}} + (zx)^{\frac{1}{2}} = 1\},$

where (and hereafter) we assume that $a^{\frac{1}{2}} > 0$ when $a > 0$. On rationalizing the defining relation in (1.3), we have

(1.4) $S_+ = \{P \in \boldsymbol{R}_+^3 ; 1 > xy + yz + zx, \; (1 - xy - yz - zx)^2 - 4(x + y + z)xyz - 8xyz = 0\},$

where (and hereafter) we put, for $k \subset R$, $k_+ = \{a \in k ; a > 0\}$.

For $k \subset \boldsymbol{R}$, we put

(1.5) $$S_+(k) = S_+ \cap k^3.$$

Let A, B, C be angles of $t = (a, b, c)$ so that A is between sides b and c; similarly for B, C. Call θ a map: $Tr \to \boldsymbol{R}_+^3$ given by

(1.6) $\theta(t) = (\tan^2(A/2), \tan^2(B/2), \tan^2(C/2)).$

Since θ is defined by angles only, it induces a map $\tilde{\theta} : \widetilde{Tr} \to \boldsymbol{R}_+^3$.

(1.8) **Theorem.** *For any subfield $k \subset \boldsymbol{R}$, the map $\tilde{\theta}$ induces a bijection*:

$$\widetilde{Tr}(k) \rightleftarrows S_+(k).$$

Proof. By abuse of notation, put

(1.9) $f(\alpha) = \tan\alpha, \; \alpha \in I = (0, \pi/2).$

Note that f is a monotone increasing function with range $(0, +\infty)$ which satisfies the functional equation

(1.10) $f(\alpha)f(\pi/2 - \alpha) = 1, \; \alpha \in I,$

and the (stronger form of) addition formula

(1.11) $f(\alpha)f(\beta) + f(\beta)f(\gamma) + f(\gamma)f(\alpha) = 1 \Leftrightarrow \alpha + \beta + \gamma = \pi/2.$

Now let $t = (a, b, c) \in Tr$ and A, B, C be angles of t as above. Putting $\alpha = A/2$, $\beta = B/2$, $\gamma = C/2$ in (1.9), (1.11), we find that the point $\theta(t) = (f(\alpha)^2, f(\beta)^2, f(\gamma)^2)$ belongs to S_+.

It is obvious that $\theta(t) = \theta(t')$ implies $t \sim t'$. Hence the map $\tilde{\theta} : \widetilde{Tr} \to S_+$ is injective. Next, for a subfield $k \subset \boldsymbol{R}$, let $t = (a, b, c) \in Tr(k)$. Then $\cos A = (b^2 + c^2 - a^2)/2bc$ belongs to k and so does $f(\alpha)^2 = (1 - \cos A)/(1 + \cos A)$; similarly for $f(\beta)^2$, $f(\gamma)^2$. Hence $\tilde{\theta}$ induces an injection $\widetilde{Tr}(k) \to S_+(k)$. Finally, it remains to show that this map is surjective. So take any point $P = (x, y, z) \in S_+(k)$. By (1.11), we can find angles A, B, C, $0 < A, B, C < \pi$ so that $A + B + C = \pi$ and that $x = f(\alpha)^2$, $y = f(\beta)^2$, $z = f(\gamma)^2$, where $\alpha = A/2$, etc. Choose a triangle $t = (a, b, c) \in Tr$ with angles A, B, C such that $c = 1$. (In case t happens to be a right triangle, we may assume without loss of generality that $C = \pi/2$, i.e., $c =$ the hypotenuse of $t = 1$.) Note that $\cos A = (1 - f(\alpha)^2)/(1 + f(\alpha)^2) = (1 - x)/(1 + x) \in k$; similarly $\cos B$, $\cos C \in k$. On the other hand, though $\sin A = 2f(\alpha)/(1 + f(\alpha)^2)$ may *not* belong to k in general, note also that $\sin^2 A = 4x/(1 + x)^2 \in k$; similarly for $\sin^2 B$, $\sin^2 C$. On squaring each term of the sine formula, we have

(1.11) $a^2/\sin^2 A = b^2/\sin^2 B = 1/\sin^2 C,$

so we see that a^2, b^2 belong to k. Since $\cos A$, $\cos B$ are both non-zero elements of k (by our assumption on the angle C), the cosine formulas