On an Algebra of a Certain Class of Operators in a Slab Domain in R^2

By Hideo UEDA

Department of Mathematics, Osaka University

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Abstract: In this paper a Dirichlet problem for the Laplacian in a domain with a corner in $\boldsymbol{R}^2$ is treated and a new method of construction of a local parametrix for that Dirichlet problem at a corner point is given. In order to construct a parametrix a certain class of operators in a slab domain is introduced and it is shown that that operator class is an algebra.

Key words: Dirichlet problems; Domains with a corner; Parametrices; Pseudo-differential operators.

1. Introduction. Let Ω be a domain with a corner in $\boldsymbol{R}^2$ given by

$$\Omega = \{(t\cos\theta,\ t\sin\theta)\ ;\ t \in \boldsymbol{R}_+,\ \varphi_1(t) < \theta < \varphi_2(t)\},$$

where $\varphi_1(t)$, $\varphi_2(t)$ are C^∞ functions on $\bar{\boldsymbol{R}}_+$ and $0 \le \varphi_1(t) < \varphi_2(t) < 2\pi$, $t \in \bar{\boldsymbol{R}}_+$, and $\varphi_1(0) = 0$, $\varphi_2(0) = \alpha$ and any derivative of $\varphi_j(e^x)$, $j = 1,2$, is bounded on $\boldsymbol{R}$. Let

$$\Gamma_j = \{(t\cos\varphi_j(t),\ t\sin\varphi_j(t))\ ;\ t \in \boldsymbol{R}_+\},\ j = 1,2.$$

We consider the following Dirichlet problem:

$$\text{(D)} \qquad \begin{cases} \Delta u = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_1, \\ u = 0 \text{ on } \Gamma_2, \end{cases}$$

where f is a given function.

We consider (D) in polar coordinates and we map Ω into $\boldsymbol{R}_+ \times (0, \alpha)$ by an appropriate coordinate transformation. Moreover, by the change of variable $t = e^x$, we map $\boldsymbol{R}_+ \times (0, \alpha)$ into $\boldsymbol{R} \times (0, \alpha)$. Then we have the following Dirichlet problem ($\tilde{\mathrm{D}}$):

$$(\tilde{\mathrm{D}}) \qquad \begin{cases} Lw = g \text{ in } \boldsymbol{R} \times (0, \alpha), \\ w = 0 \text{ on } \boldsymbol{R} \times \{0\}, \\ w = 0 \text{ on } \boldsymbol{R} \times \{\alpha\}, \end{cases}$$

where L is a strongly elliptic operator of second order in $\boldsymbol{R} \times [0, \alpha]$ and coefficients of L are real valued C^∞ functions in $\boldsymbol{R} \times [0, \alpha]$ whose derivatives of any order are bounded in $\boldsymbol{R} \times [0, \alpha]$ and the principal part of L is written in the form

$$L_0 = \partial_x{}^2 + 2a_1(x,\theta)\partial_x\partial_\theta + a_2(x,\theta)\partial_\theta{}^2.$$

Since L is a strongly elliptic operator in $\boldsymbol{R} \times [0, \alpha]$ and coefficients of L_0 are real valued functions in $\boldsymbol{R} \times [0, \alpha]$, there exists a constant $\delta > 0$ such that

$$(1.1) \qquad a_2(x,\theta) - a_1(x,\theta)^2 \ge \delta\ ; \quad x \in \boldsymbol{R},\ \theta \in [0, \alpha].$$

The purpose of this paper is to construct a global parametrix for the problem ($\tilde{\mathrm{D}}$). For constructing a parametrix, we shall introduce a class of operators expressed by a sum of two integrals in two parameters in $[0, \alpha]$ of pseudo-differential operators on $\boldsymbol{R}$. This class of operators is closed under taking products of operators and taking formal adjoints. Those properties play a crucial role in the construction of a parametrix. Since we use only the condition (1.1) for constructing a parametrix, our method is applicable to general operators which have strong ellipticity instead of the Laplacian. Our parametrix is concerned with the Mellin transform and solutions of Dirichlet problems for the Laplacian in wedges in $\boldsymbol{R}^2$, cf. [1] and[7].

All the lemmas and theorems are stated without proofs.

2. A class of operators and its algebra. We shall use the following notations:

$\mathcal{S}(\boldsymbol{R})$ denotes the set of all rapidly decreasing functions on $\boldsymbol{R}$. For $w \in \mathcal{S}(\boldsymbol{R})\, \hat{w}$ denotes the Fourier transform of w. For $\xi \in \boldsymbol{R}$ we write $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. We say that $a(x,\xi) \in S^m_{1,0} (m \in \boldsymbol{R})$ when $a(x,\xi) \in C^\infty(\boldsymbol{R}^2)$ and for any γ_1, $\gamma_2 \ge 0$ there exists a constant $C_{\gamma_1,\gamma_2} > 0$ such that

$$|\partial_x^{\gamma_1}\partial_\xi^{\gamma_2} a(x,\xi)| \le C_{\gamma_1,\gamma_2}\langle\xi\rangle^{m-\gamma_2}; \quad x,\ \xi \in \boldsymbol{R}.$$

We set $S^\infty_{1,0} = \cup_m S^m_{1,0}$. For $a(x,\xi) \in S^\infty_{1,0}$ we define a pseudo-differential operator $a(X, D_x)$