Gamelin Constants of Two-sheeted Discs

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For any $0 < \delta < 1$ and n, an n-tuple $\{f_j\}$ of functions $f_1, \ldots, f_n$ in the family $H^\infty(R)$ of bounded holomorphic functions on a Riemann surface R is referred to as a corona datum of index (n, δ) if the following condition is satisfied:

(1) $$\delta \leq (\textstyle\sum_j |f_j|^2)^{1/2} \leq 1.$$

An n-tuple $\{g_j\}$ of functions $g_1, \ldots, g_n$ in $H^\infty(R)$ is said to be a corona solution of the datum $\{f_j\}$ if $\sum_j f_j g_j = 1$. The quantity $C(R\,;n, \delta)$ given by

(2) $$C(R\,;n,\delta) = \sup_{\{f_j\}}(\inf_{\{g_j\}}(\sup_{p\in R}(\textstyle\sum_j |g_j(p)|^2)^{1/2}))$$

will be referred to as the Gamelin constant of R of index (n, δ) where the first supremum is taken with respect to corona data $\{f_j\}$ of index (n, δ) on R and the infimum is taken with respect to corona solutions $\{g_j\}$ of each fixed datum $\{f_j\}$ under the usual convention that $\inf_{\{g_j\}} = \infty$ if there exist no corona solutions $\{g_j\}$ of the datum $\{f_j\}$.

We assume that R is a two-sheeted unlimited covering surface over the unit disc D, which we call a two-sheeted disc. We will show the following

Theorem 1. *For each $0 < \delta < 1$, there exists a constant $C(\delta)$ depending only on δ such that*

(3) $$C(\delta) = \sup_n(\sup_R C(R\,;n,\delta)) < \infty,$$

where n runs over all positive integers and R runs over all two-sheeted discs.

Corollary. *Let R be any two-sheeted disc. Let $\{f_j\}$ be a sequence of functions in $H^\infty(R)$ such that $0 < \delta \leq (\sum_j |f_j|^2)^{1/2} \leq 1$. Then there exists a sequence of functions $\{g_j\}$ in $H^\infty(R)$ and a constant $c(\delta)$ depending only on δ such that $\sum_j f_j g_j = 1$ and $(\sum_j |g_j|^2)^{1/2} \leq c(\delta)$.*

Let (R, π, D) be any two-sheeted disc with projection π. For any f in $H^\infty(D)$, the function $f \cdot \pi$ belongs to $H^\infty(R)$. We identify f with $f \cdot \pi$, so that $H^\infty(D)$ is a subset of $H^\infty(R)$. If R has too many branch points, it holds that $H^\infty(R) = H^\infty(D)$, where Corollary was proved by M. Rosenblum [5] and V. A. Tolokonnikov [6] (cf. [4]).

1. In order to prove Theorem 1, by a normal families argument it is enough to show the following

Theorem 2. *Let R be a two-sheeted disc defined by a two-valued function $\zeta = \sqrt{B}$, where B is a finite Blaschke product whose zeros are all simple. If an n-tuple of*

(4) $$f_j = a_j + b_j\sqrt{B} \quad (j = 1, \ldots, n)$$

is a corona datum of index (n, δ) on R such that a_j and b_j are holomorphic on some neighbourhood of $\bar{D}$, then there exists a corona solution $\{g_j\}$ of $\{f_j\}$ such that

$$(\textstyle\sum_j |g_j|^2)^{1/2} \leq C\delta^{-12},$$

where C is a constant independent of δ, B and n.

We will prove Theorem 2 in §§.2-7. In §.2 we introduce a function ρ, which plays an important role in our proof. In §§.3 and 4 corona solutions are given. By duality, those estimates are reduced to ones of four functions, which are accomplished in §§.5 and 6. Our proof is concluded in §.7.

2. Let $(\cdot,\cdot)$ and $\|\cdot\|$ be the inner product and norm of C^n. Let $a = (a_1, \cdots, a_n)$, $b = (b_1, \cdots, b_n)$ and $f = (f_1, \cdots, f_n)$,

(5) $$\rho = \|a\|^4 + \|b\|^4|B|^2 - (a,b)^2\bar{B} - (b,a)^2 B + (\|a\|^2\|b\|^2 - |(a,b)|^2)(|B|^2 + 1),$$

(6) $$x_j = (\|a\|^2 + \|b\|^2)a_j - \{(a,b) + (b,a)B\}b_j$$

and

(7) $$y_j = -\{(a,b) + (b,a)B\}a_j + (\|a\|^2 + \|b\|^2)Bb_j.$$

Proposition 1. *ρ, x_j and y_j are smooth on some neighbourhood of $\bar{D}$ such that $\rho \geq \delta^4$ and $\sum_j(a_j + b_j\sqrt{B})(\bar{x}_j + \bar{y}_j\sqrt{B}) = \rho$.*

Proof. By (1) and (4), we have $\sum_j |a_j + b_j\sqrt{B}|^2 \geq \delta^2$ and $\sum_j |a_j - b_j\sqrt{B}|^2 \geq \delta^2$. Since $2|B| \leq |B|^2 + 1$ and

$$(\textstyle\sum_j |a_j + b_j\sqrt{B}|^2)(\sum_j |a_j - b_j\sqrt{B}|^2)$$
$$= \|a\|^4 + \|b\|^4|B|^2 - (a,b)^2\bar{B} - (b,a)^2 B + 2(\|a\|^2 + \|b\|^2 - |(a,b)|^2)|B|,$$

we obtain $\rho \geq \delta^4$. □

We may assume that functions x_j and y_j are smooth and have compact supports in the com-