Best Constant in Weighted Sobolev Inequality*)

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1. Introduction and results. The purpose of the present paper is to study the best constant in the imbedding theorems for the weighted Sobolev spaces with weight functions being powers of $|x|$. We shall deal with the weighted Sobolev spaces denoted by $W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ and $R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$, where p, n, α, β satisfy $n \geq 2$, $1 < p < n/(1-\alpha+\beta)$ and $\alpha, \beta > -n/p$ (See also (1.5)). Let $L^p_\alpha(\boldsymbol{R}^n)$ denote the space of Lebesgue measurable functions, defined on $\boldsymbol{R}^n$, for which

(1.1) $$\| u ; L^p_\alpha(\boldsymbol{R}^n) \| = \left(\int_{\boldsymbol{R}^n} |u|^p |x|^{\alpha p} dx \right)^{1/p} < +\infty.$$

$W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ is defined as the completion of $C^\infty_0(\boldsymbol{R}^n)$ with respect to the norm

(1.2) $$\| u ; W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n) \| = \| u ; L^q_\beta(\boldsymbol{R}^n) \| + \| |\nabla u| ; L^p_\alpha(\boldsymbol{R}^n) \|,$$

where $q = q(p, \alpha, \beta, n)$ is the so-called Sobolev exponent defined by

(1.3) $$q = q(p, \alpha, \beta, n) \equiv \frac{np}{n - p(1-\alpha+\beta)}.$$

Here we note that q satisfies the equality in (1.5), and if $\alpha = \beta$ then q equals $np/(n-p)$, $R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ is defined as

(1.4) $$R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n) = \{u \in W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n) ; u \text{ is a radial function}\}.$$

We shall study the following variational problems. Assume that p, q, n, α and β satisfy

(1.5) $$\begin{cases} 1 < p < +\infty, \ (1-\alpha+\beta)p < n, & n \geq 2, \\ 0 < 1/p - 1/q = (1-\alpha+\beta)/n \end{cases}$$

and

(1.6) $$-n/q < \beta \leq \alpha.$$

Under these assumptions (1.5) and (1.6), we set

(P) $$S(p, q, \alpha, \beta, n) = \inf\left[\int_{\boldsymbol{R}^n} |\nabla u|^p |x|^{p\alpha} dx ; \ u \in W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n), \| u ; L^q_\beta(\boldsymbol{R}^n) \| = 1 \right].$$

In the following problem (P_R), we assume instead of the inequality (1.6)

(1.7) $$-n/q < \beta.$$

Under the assumptions (1.5) and (1.7), we set

(P_R) $$S_R(p, q, \alpha, \beta, n) = \inf\left[\int_{\boldsymbol{R}^n} |\nabla u|^p |x|^{p\alpha} dx ; \ u \in R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n), \| u ; L^q_\beta(\boldsymbol{R}^n) \| = 1 \right].$$

By a suitable change of variables this variational problem (P_R) in the radial space $R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ is reduced to prove the classical Sobolev inequality, which was solved by G. Talenti using the notion of Hilbert invariant integral (Lemma 2 in [12]), and the infimum is achieved by functions of the form

(1.8) $$v(x) = [a + b|x|^{\frac{hp}{p-1}}]^{1-\frac{n}{p(1-\alpha+\beta)}}, \quad h = \frac{(1-\alpha+\beta)(n-p+p\alpha)}{n - p(1-\alpha+\beta)}.$$

Then with somewhat more calculations we see

Lemma 1.1. *Assume that* (1.5) *and* (1.7). *Then we have*

(1.9) $S_R(p, q, \alpha, \beta, n) = I_R(p, q, \alpha, \beta, n)$, *where*

(1.10) $$I_R(p, q, \alpha, \beta, n) = \pi^{\frac{p\gamma}{2}} \cdot n \cdot \left(\frac{n-\gamma p}{p-1} \right)^{p-1} \cdot \left(\frac{n-p+p\alpha}{n-\gamma p} \right)^{p-\frac{p\gamma}{n}} \cdot \left(\frac{2(p-1)}{\gamma p} \right)^{\frac{p\gamma}{n}} \times \left\{ \frac{\Gamma(n/\gamma p)\Gamma(n(p-1)/\gamma p)}{\Gamma(n/2)\Gamma(n/\gamma)} \right\}^{\frac{p\gamma}{n}},$$

where $\gamma = 1 - \alpha + \beta$. *In particular if* $\alpha = \beta$, *then we have*

(1.11) $$S_R(p, q, \alpha, \alpha, n) = S(p, q, n) \cdot \left(\frac{n-p+p\alpha}{n-p} \right)^{p-\frac{p}{n}},$$

where we set $S(p, q, n) = S(p, q, 0, 0, n)$ *conventionally.*

Therefore we immediately get

Lemma 1.2. *Assume that* $1/p - 1/q = 1/n$, $1 < p < n$ *and* $n > 2$. *If* $\alpha > 0$ [*respectively* $\alpha < 0$], *then it holds that* $S(p, q, n) < S_R(p, q, \alpha, \alpha, n)$ [*respectively* $S(p, q, n) > S_R(p, q, \alpha, \alpha, n)$]. *Here* $S(p, q, n) = S(p, q, 0, 0, n)$ *as in* (1.11).

From this lemma it seems that if $\alpha \leq 0$, $S_R(p, q, \alpha, \beta, n)$ is also the best constant for

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