On Hasse Zeta Functions of Enveloping Algebras of Solvable Lie Algebras 2

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1. Introduction. 1.1. In our previous papers [3], [4], we have studied the Hasse zeta functions $\zeta_A(s)$ for finitely generated rings A over the ring $\boldsymbol{Z}$ of integers. These functions which have been known for commutative A, were generalized in [3] to the case where A is not necessarily commutative (concerning the definition, see 1.3 below). In [4], we studied them for the following case. Let R be a finitely generated commutative ring over $\boldsymbol{Z}$. Let $\mathfrak{g}$ be a solvable Lie algebra over R which is free of finite rank n as an R-module, and let A be the universal enveloping algebra of $\mathfrak{g}$ over R. In [4], we proved $\zeta_A(s) = \zeta_R(s-n)$ under certain conditions on p-mappings on $\mathfrak{g}$.

Now we shall show that these conditions can be eliminated, that is, we have

Theorem 1.2. *Let R, $\mathfrak{g}$, n, and A be as above. Then*

$$\zeta_A(s) = \zeta_R(s-n).$$

Theorem 1.2 follows from Theorem 1.4 below. Before stating it, we shall first review the definition of the function $\zeta_A(s)$.

1.3. For a (not necessarily commutative) finitely generated ring A over $\boldsymbol{Z}$, the Hasse zeta function $\zeta_A(s)$ of A is defined by

$$\zeta_A(s) = \prod_{r\geq 1} \zeta_{A,r}(s)$$

where r runs over integers ≥ 1 and,

$$\zeta_{A,r}(s) = \prod_{p} \exp \sum_{n=1}^{\infty} \frac{\# \mathfrak{S}_{A,r}(\boldsymbol{F}_{p^n})}{n} (p^{-s})^n$$

where $\mathfrak{S}_{A,r}$ is a certain scheme of finite type over $\boldsymbol{Z}$, p runs over prime numbers, and $\boldsymbol{F}_{p^n}$ is a finite field with p^n elements, so the function $\zeta_{A,r}(s)$ coincides with the product of Weil's zeta functions of $\mathfrak{S}_{A,r} \otimes_{\boldsymbol{Z}} \boldsymbol{F}_p$ [2] for all prime numbers p. For the algebraic closure K of $\boldsymbol{F}_p$, $\mathfrak{S}_{A,r}(K)$ is identified with the set of the isomorphism classes of all r-dimensional irreducible representations of A over K, and $\mathfrak{S}_{A,r}(\boldsymbol{F}_{p^n})$ is identified with the $\mathrm{Gal}(K/\boldsymbol{F}_{p^n})$-fixed part of $\mathfrak{S}_{A,r}(K)$.

We may assume that R is a finite field of characteristic $p>0$, for $\zeta_R(s)$, $\zeta_A(s)$ are products of $\zeta_{R/\mathfrak{m}}(s)$, $\zeta_{A/\mathfrak{m}A}(s)$ over all maximal ideals $\mathfrak{m}$ of R, respectively. So assume R is a finite field k of characteristic p.

Theorem 1.4. *Let k be a finite field. Let B be a finitely generated algebra over k. Let δ be a k-derivation of B, and let A be the ring $\{\sum_{i=0}^{N} b_i t^i$; $N \geq 0$, $b_i \in B\}$ in which t is an indeterminate and the multiplication is given by $tb - bt = \delta(b)$ $(b \in B)$. Then*

$$\zeta_A(s) = \zeta_B(s-1).$$

As we may assume that the ring R is a finite field k, and as $\mathfrak{g}$ is a solvable Lie algebra, there exists a sequence of subalgebras of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_n = \{1\}$$

where $\mathfrak{g}_i$ is of dimension $n-i$ as a k-vector space of $\mathfrak{g}$, and $[\mathfrak{g}_{i-1}, \mathfrak{g}_i] \subset \mathfrak{g}_i$ for $1 \leq i \leq n$. Take the universal enveloping algebras of $\mathfrak{g}_{i-1}$ and $\mathfrak{g}_i$ as A and B, respectively, and apply Theorem 1.4 inductively, then we obtain Theorem 1.2. Hence it is sufficient to prove Theorem 1.4. But as the proof of Theorem 1.4 is complicated, we shall give here its proof only for the case that B is commutative, leaving the general proof for another publication.

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2. Proof of Theorem 1.4 (in the case that B is commutative). **2.1.** Let k, A, and B be as in the assumption of Theorem 1.4. As described before, in this section, let B be furthermore commutative. Let K be the algebraic closure of k.

Let $\mathfrak{S}_A = \coprod_{r\geq 1} \mathfrak{S}_{A,r}$, and for an extension k' of k, let $\mathfrak{S}_A^k(k')$ be the set of k'-rational points of $\mathfrak{S}_A$ as a k-scheme. Let $\mathfrak{S}_B = \mathrm{Spec}(B)$. We define $\mathfrak{S}_B^k(k')$ as we defined $\mathfrak{S}_A^k(k')$. It is sufficient to show that as a $\mathrm{Gal}(K/k)$-set (i.e. as a set endowed with an action of $\mathrm{Gal}(K/k)$),

$$\mathfrak{S}_A^k(K) \simeq \mathfrak{S}_B^k(K) \times K.$$

2.2. Let M be a finite dimensional irreduci-