A Remark on Estimates of Bilinear Forms of Gradients in Hardy Space

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§0. Introduction. In a recent interesting paper [1] L.C. Evans and S. Müller established the estimate of local Hardy space norm of gradients ϕ_{x_1}, ϕ_{x_2}:

$$\text{(0.1)}\quad \|\phi\phi_{x_1}\phi_{x_2}\|_{h^1} + \|\phi(\phi_{x_1}^2 - \phi_{x_2}^2)\|_{h^1} \le C(\|\phi_{x_1}\|^2_{L^2(B(0,R))} + \|\phi_{x_2}\|^2_{L^2(B(0,R))})$$

provided that

$$\text{(0.2)}\quad -\Delta\psi = \omega \geqq 0 \text{ in } \boldsymbol{R}^2.$$

Here ϕ is in $C_0^\infty(\boldsymbol{R}^2)$ and the constants C and R depends only on ϕ ; h^1 is a local Hardy space defined in §1 and $B(x, R)$ denotes the closed ball of radius R centered at $x \in \boldsymbol{R}^2$. (Another proof based on harmonic analysis is given by Semmes [2].)

This estimate is useful in proving the existence of weak solutions for the initial value problem of the two-dimensional Euler equation when the vorticity of the initial value is nonnegative measure ([1] and Delort [3]). The assumption $\omega \geqq 0$ in (0.2) is essential for the estimate (0.1); in fact, Evans and Müller [1] gave a counterexample for (0.1) when the condition $\omega \geqq 0$ is violated. However, in their example the set where ω is nonnegative may be complicated.

In this paper we give another counterexample for (0.1) even when ω is odd in the second variable x_2 i.e. $\omega(x_1, x_2) = -\omega(x_1, -x_2)$ and $\omega(x_1, x_2) \geqq 0$ for $x_2 \geqq 0$. This suggests that it is difficult to extend weak solutions for the initial-boundary value problem of the Euler equation when the domain is a half space $\boldsymbol{R}^2_+$ even if initial value is nonnegative in $\boldsymbol{R}^2_+$.

To get our counterexample we construct a sequence ψ^ε of form $\psi^\varepsilon(x) = \psi(x/\varepsilon)$. A key observation is the existence of function ψ that satisfies

$$\int_{\boldsymbol{R}^2} \psi_{x_1}^2\, dx \neq \int_{\boldsymbol{R}^2} \psi_{x_2}^2\, dx$$

with $-\Delta\psi = \omega$, where $\omega \in C_0^\infty(\boldsymbol{R}^2)$ is odd in the second variable x_2 and $\omega \geqq 0$ in $\boldsymbol{R}^2_+$, and $\psi \in H^1(\boldsymbol{R}^2)$; $H^1(\boldsymbol{R}^2)$ denotes the Sobolev space, i.e. the space of $f \in L^2(\boldsymbol{R}^2)$ with $f_{x_1}, f_{x_2} \in L^2(\boldsymbol{R}^2)$.

§1. Definition and main theorem. We begin with definition of local Hardy space as in [1].

Definition 1.1. Let η be in $C_0^\infty(\boldsymbol{R}^n)$ with $\operatorname{supp}\eta \subset B(0,1)$ and $\int_{\boldsymbol{R}^n} \eta\, dx = 1$. For a function f in $L^1_{loc}(\boldsymbol{R}^n)$, f^{**} is defined by

$$\text{(1.1)}\quad f^{**}(x) = \sup_{0<r<1} \left| r^{-n} \int_{\boldsymbol{R}^n} \eta\left(\frac{x-y}{r}\right) f(y)\, dy \right|.$$

The local Hardy space $\mathscr{H}^1_{loc}$ is defined by

$$\text{(1.2)}\quad \mathscr{H}^1_{loc}(\boldsymbol{R}^n) = \{ f \in L^1_{loc}(\boldsymbol{R}^n) \mid f^{**} \in L^1_{loc}(\boldsymbol{R}^n) \}.$$

We recall the normed local Hardy space h^1 defined by

$$\text{(1.3)}\quad h^1(\boldsymbol{R}^n) = \{ f \in L^1(\boldsymbol{R}^n) \mid f^{**} \in L^1(\boldsymbol{R}^n) \}$$

with the norm

$$\|f\|_{h^1(\boldsymbol{R}^n)} = \|f^{**}\|_{L^1(\boldsymbol{R}^n)}.$$

Definition 1.2. For a function f in $C_0^\infty(\boldsymbol{R}^2)$, we define the operator $(-\Delta)^{-1}$ by

$$\text{(1.4)}\quad (-\Delta)^{-1} f(x) = \frac{1}{2\pi} \int_{\boldsymbol{R}^2} f(y) \log|x-y|\, dy.$$

Theorem 1.3. *Let T and S be the spaces of form*

$$T = \{ \omega \in C_0^\infty(\boldsymbol{R}^2) \mid \omega(x_1, x_2) \geqq 0 \text{ for } x_2 \geqq 0,\ \omega(x_1, x_2) = -\omega(x_1, -x_2) \},$$
$$S = \{ (-\Delta)^{-1}\omega \mid \omega \in T \}.$$

Then there exists a sequence $\{\psi^\varepsilon\}_{0<\varepsilon<1}$ in S such that

$$\sup_{0<\varepsilon<1} \|\psi^\varepsilon\|_{H^1(\boldsymbol{R}^2)} < \infty$$

and

$$\text{(1.5)}\quad \lim_{\varepsilon \downarrow 0} \|\phi(\psi^\varepsilon_{x_1})^2 - (\psi^\varepsilon_{x_2})^2\}\|_{h^1(\boldsymbol{R}^2)} = \infty$$

where $\phi \in C_0^\infty(\boldsymbol{R}^2)$, $0 \leqq \phi \leqq 1$, $\phi|_{B(0,1/8)} \equiv 1$ *and* $\operatorname{supp}\phi \subset B(0,1/2)$.

§2. Proof of theorem. At first, we show a fundamental estimate in normed local Hardy space; this is an extension of a result to Evans and Müller [1].

Lemma 2.1. *Assume that f is in $L^1(\boldsymbol{R}^n)$, and $\int_{\boldsymbol{R}^n} f(x)\, dx = C_f \neq 0$. Let $f^\varepsilon(x) = \frac{1}{\varepsilon^n} f\left(\frac{x}{\varepsilon}\right)$.*