Algorithms for b-Functions, Induced Systems, and Algebraic Local Cohomology of D-Modules

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1. Introduction. Let K be an algebraically closed field of characteristic zero and let X be a Zariski open set of K^n with a positive integer n. We fix a coordinate system $x = (x_1, \dots, x_n)$ of X and write $\partial = (\partial_1, \dots, \partial_n)$ with $\partial_i := \partial / \partial x_i$. We denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators on X (cf. [2], [3]).

We assume that (a presentation of) a coherent left $\mathcal{D}_X$-module $\mathcal{M}$ is given. Let u be a section of $\mathcal{M}$ and let $f = f(x)$ be an arbitrary polynomial of n variables. Let s be an indeterminate. If $\mathcal{M}$ is holonomic, then for each point p of $Y := \{x \in X \mid f(x) = 0\}$, there exist a germ $P(x, \partial, s)$ of $\mathcal{D}_X[s]$ at p and a polynomial $b(s) \in K[s]$ of one variable so that

(1.1) $$P(x, \partial, s)(f^{s+1}u) = b(s) f^s u$$

holds (cf. [11]). More precisely, (1.1) means that there exists a nonnegative integer m so that

$$Q := f^{m-s}(b(s) - P(x, \partial, s) f) f^s \in \mathcal{D}_X[s]$$

satisfies $Qu = 0$ in $\mathcal{M}[s] := K[s] \otimes_K \mathcal{M}$. A monic polynomial $b(s)$ of the least degree that satisfies (1.1) is called the (generalized) b-function for f and u. When $\mathcal{M}$ coincides with the sheaf $\mathcal{O}_X$ of regular functions and $u = 1$, we get the classical b-function (or the Bernstein-Sato polynomial) of f. Algorithms for computing the Bernstein-Sato polynomial have been given by several authors ([21], [25], [4], [16]) but not for an arbitrary f.

One of the main purposes of the present paper is to give algorithms for computing the b-function for u and f and for computing the algebraic local cohomology groups $\mathcal{H}^j_{[Y]}(\mathcal{M})$ $(j = 0,1)$ as left $\mathcal{D}_X$-modules (cf. [11] for the definition). The algorithm for the local cohomology groups needs some information on the b-function.

These algorithms are actually obtained as byproducts of the solution of more general problems as follows:

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_{K \times X}$-module. For the sake of simplicity, let us assume here that a section u of $\mathcal{M}$ generates $\mathcal{M}$. We identify X with the subset $\{(t, x) \in K \times X \mid t = 0\}$ of $K \times X$. Then the b-function of u along X at $p \in X$ is a nonzero polynomial $b(s) \in K[s]$ of the least degree that satisfies

$$(b(t\partial_t) + tP(t, x, t\partial_t, \partial))u = 0$$

with a germ $P(t, x, t\partial_t, \partial)$ of $\mathcal{D}_{K \times X}$ at p, where we write $\partial_t := \partial / \partial t$. $\mathcal{M}$ is called *specializable* along X at p if such $b(s)$ exists.

We first present an algorithm which computes $b(s)$, or determines that there is none, by using a kind of Gröbner basis for the Weyl algebra related to a filtration introduced by Kashiwara [12]. Such Gröbner bases were used in [18], [19], [20].

If $\mathcal{M}$ is specializable, then its induced system to X is the complex of left $\mathcal{D}_X$-modules $\mathcal{M}^{\cdot}_X$ whose cohomology groups are coherent $\mathcal{D}_X$-modules. We also obtain an algorithm of computing the cohomology groups of $\mathcal{M}^{\cdot}_X$ by using an FW-Gröbner basis. These algorithms for the b-function and the induced system, combined with a viewpoint of Malgrange [17], provide algorithms for the b-function for a polynomial (and a section of a holonomic system), and for the algebraic local cohomology groups.

When K coincides with the field $\boldsymbol{C}$ of complex numbers, we can consider the problems explained so far with $\mathcal{D}_X$ replaced by the sheaf $\mathcal{D}_X^{\mathrm{an}}$ of *analytic* differential operators. Then our algorithms yield correct solutions also in this analytic case if the left $\mathcal{D}_X^{\mathrm{an}}$-module $\mathcal{M}^{\mathrm{an}}$ in question is written in the form $\mathcal{M}^{\mathrm{an}} = \mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{M}$ with a coherent $\mathcal{D}_X$-module $\mathcal{M}$ whose presentation is given explicitly.

We have implemented the algorithms by using a computer algebra system Kan [24]. Details of the present paper will appear elsewhere.

2. Gröbner bases. Let us denote by A_n and by A_{n+1} the Weyl algebra on n variables x, and the Weyl algebra on $n + 1$ variables (t, x) re-