A Generalization of Rosenhain's Normal Form for Hyperelliptic Curves with an Application

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Introduction. Let C be a compact Riemann surface of genus 2. Then C has six Wierstrass points. If we normalize three of them into 0, 1 and ∞, the complex curve C is defined by

$$Y^2 = X(X-1)(X-\lambda_1)(X-\lambda_2)(X-\lambda_3).$$

Rosenhain's normal form gives λ_1, λ_2 and λ_3 as ratios of theta constants at the period matrix of C (see Remark 1.3).

In this paper, we will give a similar formula for the hyperelliptic curves over $\boldsymbol{C}$ of general genus (Theorem 1.1). As an application of the formula, we will give resolutions of a complex algebraic equation as ratios of theta constants at the period matrix of a suitable hyperelliptic curve (Theorem 3.1).

Such formulas were given by H.Umemura in [1] based on Thomae's formula. But adding to Thomae's formula, we have Frobenius' theta formula [1, Theorem 7.1] and a criterion of vanishing of theta constant at the period matrix of the hyperelliptic curve [1, Corollary 6.7]. Using these results, we can simplify the formula given by Umemura.

§1 Main result. Let $f(X)$ be a separable monic polynomial with complex coefficients of degree $2g+1$. Let $a_1, a_2, \cdots, a_{2g+1}$ be the roots of $f(X)=0$. Let $\Omega \in \mathfrak{H}_g$ be the period matrix of the hyperelliptic curve $Y^2 = f(X)$. Here $\mathfrak{H}_g$ denotes the Siegel upper half space of genus g. The ordering of the roots of $f(X)=0$ determines the classical basis of the first cohomology group of the hyperelliptic curve. The basis in turn defines the period matrix Ω. A theta function is defined by

$$\vartheta[\alpha](\Omega, w) = \sum_{l\in \boldsymbol{Z}^g} \exp 2\pi\sqrt{-1} \left\{\frac{1}{2}\langle l+\alpha', (l+\alpha')\Omega\rangle + \langle l+\alpha', w+\alpha''\rangle\right\},$$

where $w \in \boldsymbol{C}^g$ and $\alpha = (\alpha', \alpha'') \in \boldsymbol{R}^{2g}$ are row vectors with $\alpha', \alpha'' \in \boldsymbol{R}^g$, and $\langle x, y\rangle = x\cdot {}^t y$.

Put

$$B = \{1, 2, 3, \cdots, 2g+1\},$$
$$U = \{1, 3, 5, \cdots, 2g+1\}.$$

Define theta characteristics $\eta_k = (\eta'_k, \eta''_k) \in \frac{1}{2}\boldsymbol{Z}^{2g}$ $(k = 1,2,\cdots, 2g+1)$ by

$$\eta'_{2i-1} = \Big(0,\cdots, 0, \overset{i}{\check{\tfrac{1}{2}}}, 0,\cdots, 0\Big),$$

$$\eta''_{2i-1} = \Big(\frac{1}{2},\cdots, \frac{1}{2}, \overset{i}{\check{0}}, 0,\cdots, 0\Big),$$

$(\eta'_{2g+1} = (0, 0, \cdots, 0))$ and

$$\eta'_{2i} = \Big(0,\cdots, 0, \overset{i}{\check{\tfrac{1}{2}}}, 0,\cdots, 0\Big),$$

$$\eta''_{2i} = \Big(\frac{1}{2},\cdots, \frac{1}{2}, \overset{i}{\check{\tfrac{1}{2}}}, 0,\cdots, 0\Big).$$

For any subset T of B, put

$$\eta_T = (\eta'_T, \eta''_T) = \sum_{k\in T} \eta_k \in \frac{1}{2}\boldsymbol{Z}^{2g},$$

$(\eta_\emptyset = (0, 0, \cdots, 0))$. For any subsets S, T of B, let us denote by $S\circ T$ the symmetric difference of S, T; $S\circ T = S\cup T - S\cap T$. For the sake of the notational simplicity, let us denote by

$$\vartheta[T] = \vartheta[\eta_T](\Omega, 0)$$

the theta zero value at the period Ω with a theta characteristic η_T for any subset T of B.

Now our main result is

Theorem 1.1. *For any disjoint decomposition* $B = V \sqcup W \sqcup \{k, l, m\}$ *with* $\# V = \# W = g-1$, *we have*

$$\frac{a_k - a_l}{a_k - a_m} = \varepsilon(k; l, m) \times \left(\frac{\vartheta[U\circ(V\cup\{k, l\})]\cdot\vartheta[U\circ(W\cup\{k, l\})]}{\vartheta[U\circ(V\cup\{k, m\})]\cdot\vartheta[U\circ(W\cup\{k, m\})]}\right)^2.$$

Here

$$\varepsilon(k; l, m) = \begin{cases} 1 & \text{if } k < l, m \text{ or } l, m < k \\ -1 & \text{if } l < k < m \text{ or } m < k < l. \end{cases}$$

The proof of Theorem 1.1 will be given in the next section.