A Note on the Extremality of Teichmüller Mappings

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Introduction. For a hyperbolic Riemann surface R, we denote by $A_2(R)$ the set of all holomorphic quadratic differentials $\phi = \phi(z)dz^2$ on R, and set

$$A_2^p(R) := \left\{\phi \in A_2(R) : \|\phi\|_p := \left(\int_R \lambda_R^{2-2p} |\phi|^p\right)^{1/p} < \infty\right\} \text{ for } 1 \leq p < \infty,$$

$$A_2^\infty(R) := \{\phi \in A_2(R) : \|\phi\|_\infty := \text{ess sup}_R \lambda_R^{-2} |\phi| < \infty\},$$

where $\lambda_R = \lambda_R(z)|dz|$ is the hyperbolic metric on R with constant negative curvature -4. For simplicity, we often write $\|\phi\|_{p,E}$ instead of $\left(\int_E \lambda_R^{2-2p} |\phi|^p\right)^{1/p}$.

A quasiconformal mapping f of a Riemann surface R is called *extremal* if it has the smallest maximal dilatation in the class Q_f of all quasiconformal mappings of R which are homotopic to f relative to the border ∂R of R. An extremal mapping is called *uniquely extremal* if there are no other extremal mappings in Q_f. Hamilton, Reich and Strebel have characterized the extremality: a quasiconformal mapping f is extremal if and only if there is a sequence $\{\phi_n\}_{n=1}^\infty$ in $A_2^1(R)$, $\|\phi_n\|_1 = 1$, such that $\lim_{n\to\infty} \int_R \mu_f \phi_n = \text{ess sup}_R |\mu_f|$, where μ_f is the Beltrami coefficient of f (Strebel [10]). Such a sequence is called a *Hamilton sequence* for f, and it is said to *degenerate* if it weakly converges to 0.

A quasiconformal mapping whose Beltrami coefficient has the form $k\bar{\phi}/|\phi|$, where $0 \leq k < 1$ and $\phi \in A_2(R)\setminus\{0\}$, is called a *Teichmüller mapping* corresponding to ϕ. In the theory of extremal quasiconformal mappings, Teichmüller mappings play an important role. We know that every Teichmüller mapping corresponding to $\phi \in A_2^1(R)$ is uniquely extremal (Strebel [10]), but there are non-extremal, and extremal but not uniquely extremal Teichmüller mappings (Strebel [8]). So it is expected to find conditions for a holomorphic quadratic differential ϕ that guarantees the Teichmüller mapping corresponding to ϕ to be extremal or not. For the case R is the unit disk $\boldsymbol{D}$, some extremality theorems have been proved, for instance, Sethares [7], Reich-Strebel [6], Hayman-Reich [2] and one of the authors [3]. On the other hand, Strebel [9] has constructed an example which shows that a lift to the universal covering of an extremal Teichmüller mapping of a compact Riemann surface is not necessarily extremal, and recently McMullen [4] and one of the authors [5] have generalized this.

1. In the present paper, we prove the following:

Theorem 1. *Suppose that R is a hyperbolic Riemann surface of finite analytic type, and that $\pi : \tilde{R} \to R$ is an infinite sheeted regular (i.e. unbounded and unramified) covering from another Riemann surface $\tilde{R}$ to R which satisfies the condition:*

(*) *for any puncture a of R and any cusped neighborhood V of a, there is an integer m such that the restriction of π to any connected component of $\pi^{-1}(V)$ is at most m sheeted.*

Then for $\Psi \in A_2^\infty(R)$, $\Psi \neq 0$, and $\phi \in \cup_{1\leq p<\infty} A_2^p(\tilde{R})$, the Teichmüller mapping $f_{\pi^*\Psi}$ corresponding to the pull-back $\pi^*\Psi \in A_2^\infty(\tilde{R})$ and the Teichmüller mapping $f_{\pi^*\Psi+\phi}$ corresponding to $\pi^*\Psi + \phi \in A_2^\infty(\tilde{R})$ have the same Hamilton sequences. In particular, $f_{\pi^*\Psi}$ is extremal if and only if so is $f_{\pi^*\Psi+\phi}$.

As an application of our Theorem 1 and McMullen's theorem, we have

Corollary 1. *Let $\pi : \tilde{R} \to R$ be a covering as in Theorem 1. If, moreover, π is nonamenable, then for any $\Psi \in A_2^\infty(R)\setminus\{0\}$ and any $\phi \in A_2^p(\tilde{R})$, $1 \leq p < \infty$, any lifts to the unit disk of the Teichmüller mapping of $\tilde{R}$ corresponding to $\pi^*\Psi + \phi$ are not extremal.*

Proof. By McMullen's theorem [4], the

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